

Definitizability of normal operators on Krein spaces and their functional calculus

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Abstract: We discuss a new concept of definitizability of a normal operator on Krein spaces. For this new concept we develop a functional calculus $\phi \mapsto \phi(N)$ which is the proper analogue of $\phi \mapsto \int \phi dE$ in the Hilbert space situation.

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1 Introduction

A bounded linear operator N on a Krein space $(\mathcal{K}, [., .])$ is normal, if N commutes with its Krein space adjoint N^+ . If we write N as $A + iB$ with the selfadjoint real part $A := \operatorname{Re} N := \frac{N+N^+}{2}$ and the selfadjoint imaginary part $B := \operatorname{Im} N := \frac{N-N^+}{2i}$, then N is normal if and only if $AB = BA$. In [K] we called a normal N definitizable whenever A and B were both definitizable in the classical sense, i.e. there exist so-called definitizing polynomials $p(z), q(z) \in \mathbb{R}[z] \setminus \{0\}$ such that $[p(A)x, x] \geq 0$ and $[q(B)x, x] \geq 0$ for all $x \in \mathcal{K}$.

For such definitizable operators in [K] we could build a functional calculus in analogy to the functional calculus $\phi \mapsto \int \phi dE$ mapping the $*$ -algebra of bounded and measurable functions on $\sigma(N)$ to $B(\mathcal{H})$ in the Hilbert space case. The functional calculus in [K] can also be seen as a generalization of Heinz Langers spectral theorem on definitizable selfadjoint operators on Krein spaces; see [L], [KP]. Unfortunately, there are unsatisfactory phenomenons with this concept of definitizability in [K]. For example, it is not clear, whether for a bijective, normal definitizable N also N^{-1} definitizable.

In the present paper we choose a more general concept of definitizability. We shall say that a normal N on a Krein space \mathcal{K} is definitizable if $[p(A, B)u, u] \geq 0$ for all $u \in \mathcal{K}$ for some, so-called definitizing, $p(x, y) \in \mathbb{C}[x, y] \setminus \{0\}$ with real coefficients. Then we study the ideal I generated by all definitizing polynomials with real coefficients in $\mathbb{C}[x, y]$, and assume that I is large in the sense that it is zero-dimensional, i.e. $\dim \mathbb{C}[x, y]/I < \infty$. By the way, if N is definitizable in the sense of [K], then I is always zero-dimensional.

Using results from algebraic geometry, under the assumption that I is zero-dimensional, the variety $V(I) = \{a \in \mathbb{C}^2 : f(a) = 0 \text{ for all } f \in I\}$ is a finite set. We split this subset of \mathbb{C}^2 up as

$$V(I) = (V(I) \cap \mathbb{R}^2) \dot{\cup} (V(I) \setminus \mathbb{R}^2),$$

and interpret $V_{\mathbb{R}}(I) := V(I) \cap \mathbb{R}^2$ in the following as a subset of \mathbb{C} by consider the first entry as the real and the second entry as the imaginary part.

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By the ascending chain condition the ideal I is generated by real defining polynomials p_1, \dots, p_m . With the help of the positive semidefinite scalar products $[p_j(A, B), \cdot, \cdot]$, $j = 1, \dots, m$ and $\sum_{k=1}^m [p_k(A, B), \cdot, \cdot]$ we construct Hilbert spaces \mathcal{H}_j , $j = 1, \dots, m$ and \mathcal{H} together with bounded and injective $T_j : \mathcal{H}_j \rightarrow \mathcal{K}$ and $T : \mathcal{H} \rightarrow \mathcal{K}$. We consider $\Theta_j : (T_j T_j^+)' \rightarrow (T_j^+ T_j)'$ and $\Theta : (TT^+)' \rightarrow (T^+ T)'$ by $\Theta_j(C) := (T_j \times T_j)^{-1}(C)$ and $\Theta(C) := (T \times T)^{-1}(C)$, as studied in [KP]. Here $(T_j T_j^+)', (TT^+)' \subseteq B(\mathcal{K})$ and $(T_j^+ T_j)' \subseteq B(\mathcal{H}_j)$, $(T^+ T)' \subseteq B(\mathcal{H})$ denote the commutant of the respective operators.

The proper family \mathcal{F}_N of functions suitable for the aimed functional calculus are functions defined on

$$(\sigma(\Theta(N)) \cup V_{\mathbb{R}}(I)) \dot{\cup} (V(I) \setminus \mathbb{R}^2).$$

Moreover, the functions $\phi \in \mathcal{F}_N$ assume values in \mathbb{C} on $\sigma(\Theta(N)) \setminus V_{\mathbb{R}}(I)$ and values in a certain finite dimensional $*$ -algebras $\mathcal{A}(z)$ at $z \in V_{\mathbb{R}}(I)$ and $\mathcal{B}((\xi, \eta))$ at $(\xi, \eta) \in V(I) \setminus \mathbb{R}^2$. On $\sigma(\Theta(N)) \setminus V_{\mathbb{R}}(I)$ we assume ϕ to be bounded and measurable. Finally, $\phi \in \mathcal{F}_N$ satisfies a growth regularity condition at all w points from $V_{\mathbb{R}}(I)$ which are not isolated in $\sigma(\Theta(N)) \cup V_{\mathbb{R}}(I)$. Vaguely speaking, this growth regularity condition means that around w the function ϕ admits an approximation by a Taylor polynomial, which is determined by $\phi(w) \in \mathcal{A}(w)$. Any polynomial $s(x, y) \in \mathbb{C}[x, y]$ can be seen as a function $s_N \in \mathcal{F}_N$ in a natural way.

For each $\phi \in \mathcal{F}_N$ we will see that there exists $p \in \mathbb{C}[x, y]$ and bounded, measurable $f_1, \dots, f_m : \sigma(\Theta(N)) \cup V_{\mathbb{R}}(I) \rightarrow \mathbb{C}$ with $f_j(z) = 0$ for $z \in V_{\mathbb{R}}(I)$ such that

$$\phi(z) = p_N(z) + \sum_j f_j(z) (p_j)_N(z) \quad (1.1)$$

for all $z \in \sigma(\Theta(N)) \cup V_{\mathbb{R}}(I)$, and that $\phi((\xi, \eta)) = p_N((\xi, \eta))$ for all $(\xi, \eta) \in V(I) \setminus \mathbb{R}^2$. We then define

$$\phi(N) := p(A, B) + \sum_{k=1}^m T_k \int_{\sigma(\Theta_k(N))} f_k dE T_k^+,$$

and show that this operator does not depend on the actual decomposition (1.1) and that $\phi \mapsto \phi(N)$ is indeed a $*$ -homomorphism satisfying $\phi(N) = s(A, B)$ for $\phi = s_N$.

2 Multiple embeddings

In the present section $(\mathcal{K}, [\cdot, \cdot])$ will be a Krein space and $(\mathcal{H}, (\cdot, \cdot))$, $(\mathcal{H}_j, (\cdot, \cdot))$, $j = 1, \dots, m$, will denote Hilbert spaces. Moreover, let $T : \mathcal{H} \rightarrow \mathcal{K}$, $T_j : \mathcal{H}_j \rightarrow \mathcal{K}$ and $R_j : \mathcal{H}_j \rightarrow \mathcal{H}$ bounded, linear and injective mappings such that $TR_j = T_j$. By $T^+ : \mathcal{K} \rightarrow \mathcal{H}$ and $T_j^+ : \mathcal{K} \rightarrow \mathcal{H}_j$ we denote the respective Krein space adjoints.

If D is an operator on a Krein space, then we shall denote by D' the commutant of D , i.e. the algebra of all operators commuting with D . For a selfadjoint D this commutant is a $*$ -algebra with respect to forming adjoint operators.

For $j = 1, \dots, m$ we shall denote by $\Theta_j : (T_j T_j^+)' (\subseteq B(\mathcal{K})) \rightarrow (T_j^+ T_j)' (\subseteq B(\mathcal{H}_j))$, and by $\Theta : (TT^+)' (\subseteq B(\mathcal{K})) \rightarrow (T^+ T)' (\subseteq B(\mathcal{H}))$ the $*$ -algebra

homomorphisms mapping the identity operator to the identity operator as in Theorem 5.8 from [KP] corresponding to the mappings T_j and T :

$$\begin{aligned}\Theta_j(C_j) &= (T_j \times T_j)^{-1}(C_j) = T_j^{-1}C_jT_j, \quad C_j \in (T_jT_j^+)', \\ \Theta(C) &= (T \times T)^{-1}(C) = T^{-1}CT, \quad C \in (TT^+)'. \end{aligned} \quad (2.1)$$

We can apply Theorem 5.8 in [KP] also to the bounded linear, injective $R_j : \mathcal{H}_j \rightarrow \mathcal{H}$, and denote the corresponding $*$ -algebra homomorphisms by $\Gamma_j : (R_jR_j^*)' (\subseteq B(\mathcal{H})) \rightarrow (R_j^*R_j)' (\subseteq B(\mathcal{H}_j))$:

$$\Gamma_j(D) = (R_j \times R_j)^{-1}(D) = R_j^{-1}DR_j, \quad D \in (R_jR_j^*)'.$$

For the following note that due to $(\text{ran } T^+)^{[\perp]} = \ker T = \{0\}$ the range of T^+ is dense in \mathcal{H} .

2.1 Lemma. *For $j = 1, \dots, m$ we have $\Theta((T_jT_j^+)' \cap (TT^+)') \subseteq (R_jR_j^*)' \cap (T^+T)'$, where in fact*

$$\Theta(C)R_jR_j^* = R_j\Theta_j(C)R_j^* = R_jR_j^*\Theta(C), \quad C \in (T_jT_j^+)' \cap (TT^+)'. \quad (2.2)$$

Moreover,

$$\Theta_j(C) = \Gamma_j \circ \Theta(C), \quad C \in (T_jT_j^+)' \cap (TT^+)'. \quad (2.3)$$

Proof. According to Theorem 5.8 in [KP] we have $\Theta_j(C)T_j^+ = T_j^+C$ and $T^+C = \Theta(C)T^+$ for $C \in (T_jT_j^+)' \cap (TT^+)'$. Therefore,

$$\begin{aligned}T(R_j\Theta_j(C)R_j^*)T^+ &= T_j\Theta_j(C)T_j^+ = T_jT_j^+C \\ &= TR_jR_j^*T^+C = T(R_jR_j^*\Theta(C))T^+. \end{aligned}$$

$\ker T = \{0\}$ and the density of $\text{ran } T^+$ yield $R_j\Theta_j(C)R_j^* = R_jR_j^*\Theta(C)$. Applying this equation to C^+ and taking adjoints yields $R_j\Theta_j(C)R_j^* = \Theta(C)R_jR_j^*$. In particular, $\Theta(C) \in (R_jR_j^*)'$. Therefore, we can apply Γ_j to $\Theta(C)$ and get

$$\Gamma_j \circ \Theta(C) = R_j^{-1}T^{-1}CTR_j = T_j^{-1}CT_j = \Theta_j(C).$$

□

For the following assertion note that by (2.3) and by the fact that Γ_j is a $*$ -algebra homomorphism mapping the identity operator to the identity operator, for $j = 1, \dots, m$ we have

$$\sigma(\Theta(C)) \subseteq \sigma(\Theta_j(C)) \quad \text{for all } C \in (T_jT_j^+)' \cap (TT^+)'. \quad (2.4)$$

2.2 Corollary. *For a $j \in \{1, \dots, m\}$ let $N \in B(\mathcal{K})$ be normal such that $N \in (T_jT_j^+)' \cap (TT^+)'$. Then $\Theta(N)$ is a normal operator in the Hilbert space \mathcal{H} , and $\Theta_j(N)$ is a normal operator in the Hilbert space \mathcal{H}_j . Denoting by $E(E_j)$ the spectral measure of $\Theta(N)$ ($\Theta_j(N)$), we have $E(\Delta) \in (R_jR_j^*)' \cap (T^+T)'$ and*

$$\Gamma_j(E(\Delta)) = E_j(\Delta),$$

for all Borel subsets Δ of \mathbb{C} , where $E_j(\Delta) \in (R_j^*R_j)' \cap (T_j^+T_j)'$.

Moreover, $\int h dE \in (R_j R_j^*)' \cap (T^+ T)'$ and

$$\Gamma_j \left(\int h dE \right) = \int h dE_j$$

for any bounded and measurable $h : \sigma(\Theta(N)) \rightarrow \mathbb{C}$, where $\int h dE_j \in (R_j^* R_j)' \cap (T_j^+ T_j)'$.

Proof. The normality of $\Theta(N)$ and $\Theta_j(N)$ is clear, since Θ and Θ_j are $*$ -homomorphisms. From Lemma 2.1 we know that $\Theta(N) \in (R_j R_j^*)' \cap (T^+ T)'$. According to the well known properties of $\Theta(N)$'s spectral measure we obtain $E(\Delta) \in (R_j R_j^*)' \cap (T^+ T)'$ and, in turn, $\int h dE \in (R_j R_j^*)' \cap (T^+ T)'$. In particular, Γ_j can be applied to $E(\Delta)$ and $\int h dE$. Similarly, $\Theta_j(N) \in (T_j^+ T_j)'$ implies $E_j(\Delta), \int h dE_j \in (T_j^+ T_j)'$ for a bounded and measurable h .

Recall from Theorem 5.8 in [KP] that $\Gamma_j(D) R_j^* x = R_j^* D$ for $D \in (R_j R_j^*)'$. Hence, for $x \in \mathcal{H}$ and $y \in \mathcal{H}_j$ we have

$$(\Gamma_j(E(\Delta)) R_j^* x, y) = (R_j^* E(\Delta) x, y) = (E(\Delta) x, R_j y)$$

and, in turn,

$$\begin{aligned} \int_{\mathbb{C}} s(z, \bar{z}) d(\Gamma_j(E) R_j^* x, y) &= \int_{\mathbb{C}} s(z, \bar{z}) d(E x, R_j y) = (s(\Theta(N), \Theta(N)^*) x, R_j y) \\ &= (R_j^* s(\Theta(N), \Theta(N)^*) x, y) = (\Gamma_j(s(\Theta(N), \Theta(N)^*)) R_j^* x, y) \end{aligned}$$

for any $s(z, w) \in \mathbb{C}[z, w]$. By (2.3) and the fact, that Γ_j is a $*$ -homomorphism, we have $\Gamma_j(s(\Theta(N), \Theta(N)^*)) = s(\Theta_j(N), \Theta_j(N)^*)$. Consequently,

$$\int_{\mathbb{C}} s(z, \bar{z}) d(\Gamma_j(E) R_j^* x, y) = \int_{\mathbb{C}} s(z, \bar{z}) d(E_j R_j^* x, y).$$

Since $E(\mathbb{C} \setminus K) = 0$ and $E_j(\mathbb{C} \setminus K) = 0$ for a certain compact $K \subseteq \mathbb{C}$ and since the set of all $s(z, \bar{z})$, $s \in \mathbb{C}[z, w]$, is densely contained in $C(K)$, we obtain from the uniqueness assertion in the Riesz Representation Theorem

$$(\Gamma_j(E(\Delta)) R_j^* x, y) = (E_j(\Delta) R_j^* x, y) \quad \text{for all } x \in \mathcal{H}, y \in \mathcal{H}_j,$$

for all Borel subsets Δ of \mathbb{C} . Due to the density of $\text{ran } R_j^*$ in \mathcal{H}_j we even have $(\Gamma_j(E(\Delta)) z, y) = (E_j(\Delta) z, y)$ for all $y, z \in \mathcal{H}_j$, and in turn $\Gamma_j(E(\Delta)) = E_j(\Delta)$. Since Γ_j maps into $(R_j^* R_j)'$, we have $E_j(\Delta) \in (R_j^* R_j)'$. This yields $\int h dE_j \in (R_j^* R_j)'$ for any bounded and measurable h .

If $h : \sigma(\Theta(N)) \rightarrow \mathbb{C}$ is bounded and measurable, then by (2.4) also its restriction to $\sigma(\Theta_j(N)) = \sigma(\Gamma_j \circ \Theta(N))$ is bounded and measurable. Due to $E_j(\Delta) R_j^* = \Gamma_j(E(\Delta)) R_j^* = R_j^* E(\Delta)$, for $x \in \mathcal{H}$ and $y \in \mathcal{H}_j$ we have

$$\begin{aligned} (\Gamma_j \left(\int h dE \right) R_j^* x, y) &= (R_j^* \left(\int h dE \right) x, y) = \left(\left(\int h dE \right) x, R_j y \right) \\ &= \int h d(E x, R_j y) = \int h d(E_j R_j^* x, y) = \left(\left(\int h dE_j \right) R_j^* x, y \right). \end{aligned}$$

The density of $\text{ran } R_j^*$ yields $\Gamma_j \left(\int h dE \right) = \int h dE_j$. \square

Recall from Lemma 5.11 in [KP] the mappings $(j = 1, \dots, m)$

$$\Xi_j : B(\mathcal{H}_j) \rightarrow B(\mathcal{K}), \quad \Xi_j(D_j) = T_j D_j T_j^+,$$

and $\Xi : B(\mathcal{H}) \rightarrow B(\mathcal{K})$ with $\Xi(D) = T D T^+$. By $(j = 1, \dots, m)$

$$\Lambda_j : B(\mathcal{H}_j) \rightarrow B(\mathcal{H}), \quad \Lambda_j(D_j) = R_j D_j R_j^*,$$

we shall denote the corresponding mappings outgoing from the mappings $R_j : \mathcal{H}_j \rightarrow \mathcal{H}$. Due to $T_j = T R_j$ we have $\Xi_j = \Xi \circ \Lambda_j$.

According to Lemma 5.11 in [KP], $\Lambda_j \circ \Gamma_j(D) = D R_j R_j^*$ for $D \in (R_j R_j^*)'$. Hence, using the notation from Corollary 2.2

$$\Xi_j\left(\int h dE_j\right) = \Xi\left(\Lambda_j \circ \Gamma_j\left(\int h dE\right)\right) = \Xi(R_j R_j^* \int h dE). \quad (2.5)$$

2.3 Lemma. Assume that for $j \in \{1, \dots, m\}$ the operator $T_j T_j^+$ commutes with $T T^+$ on \mathcal{K} . Then the operators $R_j R_j^*$, $T^+ T$ commute on \mathcal{H} and $R_j^* R_j$, $T_j^+ T_j$ commute on \mathcal{H}_j . Moreover,

$$\Theta(T_j T_j^+) = R_j R_j^* T^+ T = T^+ T R_j R_j^*. \quad (2.6)$$

Proof. If $T_j T_j^+$ and $T T^+$ commute on \mathcal{K} , then

$$T(T^+ T R_j R_j^*) T^+ = T T^+ T_j T_j^+ = T_j T_j^+ T T^+ = T(R_j R_j^* T^+ T) T^+.$$

Employing T 's injectivity and the density of $\text{ran } T^+$, we see that $R_j R_j^*$ and $T^+ T$ commute. From this we derive

$$T_j^+ T_j R_j^* R_j = R_j^* (T^+ T R_j R_j^*) R_j = R_j^* (R_j R_j^* T^+ T) R_j = R_j^* R_j T_j^+ T_j.$$

(2.6) follows from

$$T^{-1} T_j T_j^+ T = T^{-1} T R_j R_j^* T^+ T = R_j R_j^* T^+ T.$$

□

3 Definitizability

In [K] we said that a normal $N \in B(\mathcal{K})$ is definitizable, if its real part $A := \frac{N+N^+}{2}$ and its imaginary part $B := \frac{N-N^*}{2i}$ are definitizable in the sense that there exist real polynomials $p, q \in \mathbb{R}[z] \setminus \{0\}$ such that $[p(A)v, v] \geq 0$ and $[q(B)v, v] \geq 0$ for all $v \in \mathcal{K}$. In the present note we will relax this condition.

3.1 Definition. For a normal $N \in B(\mathcal{K})$ we call $p(x, y) \in \mathbb{C}[x, y] \setminus \{0\}$ a definitizing polynomial for N , if

$$[p(A, B)v, v] \geq 0 \quad \text{for all } v \in \mathcal{K}. \quad (3.1)$$

where $A = \frac{N+N^+}{2}$ and $B = \frac{N-N^*}{2i}$. If such a definitizing $p \in \mathbb{C}[x, y] \setminus \{0\}$ exists, then we call N definitizable normal. ◇

Clearly, we could also write p as a polynomial of the variables N and N^+ . But because of $A = A^+$ and $B = B^+$, writing p as a polynomial of the variables A and B has some notational advantages.

3.2 Remark. According to (3.1) the operator $p(A, B) \in B(\mathcal{K})$ must be selfadjoint; i.e. $p(A, B)^+ = p^\#(A, B)$, where $p^\#(x, y) = \overline{p(\overline{x}, \overline{y})}$. Hence, $q := \frac{p_j + p_j^\#}{2}$ is real, i.e. $q(x, y) \in \mathbb{R}[x, y] \setminus \{0\}$, and satisfies $q(A, B) = p(A, B)$. Thus, we can assume that a definitizing polynomial is real. \diamond

In the present section we assume that $p_j(x, y) \in \mathbb{R}[x, y] \setminus \{0\}$, $j = 1, \dots, m$, are real, definitizing polynomial for N .

3.3 Proposition. *With the above assumptions and notation there exist Hilbert spaces $(\mathcal{H}, (\cdot, \cdot))$, $(\mathcal{H}_j, (\cdot, \cdot))$, $j = 1, \dots, m$ and bounded linear and injective operators $T : \mathcal{H} \rightarrow \mathcal{K}$, $T_j : \mathcal{H}_j \rightarrow \mathcal{K}$, such that*

$$T_j T_j^+ = p_j(A, B), \quad \text{and} \quad T T^+ = \sum_{k=1}^m T_k T_k^+ = \sum_{k=1}^m p_k(A, B).$$

Proof. Let $(\mathcal{H}_j, (\cdot, \cdot))$ be the Hilbert space completion of $\mathcal{K}/\ker p_j(A, B)$ with respect to $[p_j(A, B)\cdot, \cdot]$ and let $T_j : \mathcal{H}_j \rightarrow \mathcal{K}$ be the adjoint of the factor mapping $x \mapsto x + \ker p_j(A, B)$ of \mathcal{K} into \mathcal{H}_j . Since T_j^+ has dense range, T_j must be injective. Similarly, let $(\mathcal{H}, (\cdot, \cdot))$ be the Hilbert space completion of $\mathcal{K}/(\ker \sum_{k=1}^m p_k(A, B))$ with respect to $[(\sum_{k=1}^m p_k(A, B))\cdot, \cdot]$ and let $T : \mathcal{H} \rightarrow \mathcal{K}$ be the injective adjoint of the factor mapping of \mathcal{K} into \mathcal{H} .

From $[T T^+ x, y] = (T^+ x, T^+ y) = (x, y) = [(\sum_{k=1}^m p_k(A, B))x, y]$ and $[T_j T_j^+ x, y] = (T_j^+ x, T_j^+ y) = (x, y) = [p_j(A, B)x, y]$ for all $x, y \in \mathcal{K}$ we conclude

$$T_j T_j^+ = p_j(A, B) \quad \text{and} \quad T T^+ = \sum_{k=1}^m p_k(A, B),$$

where the operators $T_j T_j^+ = p_j(A, B)$, $j = 1, \dots, m$, pairwise commute, because A and B do. \square

Proposition 3.3 in particular yields

$$T T^+ = \sum_{k=1}^m T_k T_k^+ \tag{3.2}$$

Since for $x \in \mathcal{K}$ and $j \in \{1, \dots, m\}$ we have

$$(T^+ x, T^+ x) = [T T^+ x, x] = \sum_{k=1}^m [T_k T_k^+ x, x] = \sum_{k=1}^m (T_k^+ x, T_k^+ x) \geq (T_j^+ x, T_j^+ x),$$

one easily concludes that $T^+ x \mapsto T_j^+ x$ constitutes a well-defined, contractive linear mapping from $\text{ran } T^+$ onto $\text{ran } T_j^+$. By $(\text{ran } T^+)^\perp = \ker T = \{0\}$ and $(\text{ran } T_j^+)^\perp = \ker T_j = \{0\}$ these ranges are dense in the Hilbert spaces \mathcal{H} and \mathcal{H}_j . Hence, there is a unique bounded linear continuation of $T^+ x \mapsto T_j^+ x$ to \mathcal{H} , which has dense range in \mathcal{H}_j .

Denoting by R_j the adjoint mapping of this continuation we clearly have $T_j = TR_j$ and $\ker R_j \subseteq \ker T_j = \{0\}$. From (3.2) we conclude

$$T(I_{\mathcal{H}})T^+ = TT^+ = \sum_{k=1}^m TR_k R_k^+ T^+ = T\left(\sum_{k=1}^m R_k R_k^+\right)T^+.$$

$\ker T = \{0\}$ and the density of $\text{ran } T^+$ yield $\sum_{k=1}^m R_k R_k^* = I_{\mathcal{H}}$.

3.4 Lemma. *With the above notations and assumptions for $j = 1, \dots, m$ there exist injective contractions $R_j : \mathcal{H}_j \rightarrow \mathcal{H}$ such that $T_j = TR_j$ and $\sum_{k=1}^m R_k R_k^* = I_{\mathcal{H}}$. Moreover, we have*

$$\{N, N^+\}' = \{A, B\}' \subseteq \bigcap_{k=1, \dots, m} (T_k T_k^+)' \subseteq (TT^+)' \quad (3.3)$$

for all $j \in \{1, \dots, m\}$. Finally,

$$\begin{aligned} p_j(\Theta(A), \Theta(B)) &= R_j R_j^* \left(\sum_{k=1}^m p_k(\Theta(A), \Theta(B)) \right) \\ &= \left(\sum_{k=1}^m p_k(\Theta(A), \Theta(B)) \right) R_j R_j^*, \end{aligned} \quad (3.4)$$

and for any $u \in \mathbb{C}[x, y]$

$$p_j(A, B)u(A, B) = \Xi_j(u(\Theta_j(A), \Theta_j(B))) = \Xi(R_j R_j^* u(\Theta(A), \Theta(B))), \quad (3.5)$$

where $\Theta : (TT^+)' (\subseteq B(\mathcal{K})) \rightarrow (T^+T)' (\subseteq B(\mathcal{H}))$ is as in (2.1) and $\Xi : B(\mathcal{H}) \rightarrow B(\mathcal{K})$ with $\Xi(D) = TDT^+$.

Proof. The first part was shown above, and (3.3) is clear from Proposition 3.3.

From (2.6) and Theorem 5.8 in [KP] we get

$$\begin{aligned} p_j(\Theta(A), \Theta(B)) &= \Theta(p_j(A, B)) = \Theta(T_j T_j^+) = R_j R_j^* T^+ T = R_j R_j^* \Theta(TT^+) \\ &= R_j R_j^* \Theta\left(\sum_{k=1}^m p_k(A, B)\right) = R_j R_j^* \left(\sum_{k=1}^m p_k(\Theta(A), \Theta(B))\right), \end{aligned}$$

where $R_j R_j^*$ commutes with $T^+T = \sum_{k=1}^m p_k(\Theta(A), \Theta(B))$ by Lemma 2.3. Finally, (3.5) follows from (see Lemma 5.11 in [KP])

$$\begin{aligned} p_j(A, B)u(A, B) &= \Xi_j(\Theta_j(u(A, B))) = \Xi \circ \Lambda_j \circ \Gamma_j(\Theta(u(A, B))) \\ &= \Xi(R_j R_j^* u(\Theta(A), \Theta(B))). \end{aligned}$$

□

By (3.3) we can apply Corollary 2.2 in the present situation. In particular, $\Theta(N)$ is a normal operator on the Hilbert space \mathcal{H} . Condition (3.1) for $p = p_j$, $j = 1, \dots, m$, implies certain spectral properties of $\Theta(N)$.

3.5 Lemma. *With the above assumptions and notation for $j \in \{1, \dots, m\}$ we have*

$$\{z \in \mathbb{C} : |p_j(\text{Re } z, \text{Im } z)| > \|R_j R_j^*\| \cdot \left| \sum_{k=1}^m p_k(\text{Re } z, \text{Im } z) \right|\} \subseteq \rho(\Theta(N)).$$

In particular, the zeros of $\sum_{k=1}^m p_k(\text{Re } z, \text{Im } z)$ in \mathbb{C} are contained in $\rho(\Theta(N)) \cup \{z \in \mathbb{C} : p_j(\text{Re } z, \text{Im } z) = 0 \text{ for all } j = 1, \dots, m\}$.

Proof. Let $n \in \mathbb{N}$ and set

$$\Delta_n := \{z \in \mathbb{C} : |p_j(\operatorname{Re} z, \operatorname{Im} z)|^2 > \frac{1}{n} + \|R_j R_j^*\|^2 \cdot \left| \sum_{k=1}^m p_k(\operatorname{Re} z, \operatorname{Im} z) \right|^2\}.$$

For $x \in E(\Delta_n)(\mathcal{H})$, where E denotes $\Theta(N)$'s special measure, we then have

$$\begin{aligned} \|p_j(\Theta(A), \Theta(B))x\|^2 &= \int_{\Delta_n} |p_j(\operatorname{Re} \zeta, \operatorname{Im} \zeta)|^2 d(E(\zeta)x, x) \geq \\ &\int_{\Delta_n} \frac{1}{n} d(E(\zeta)x, x) + \|R_j R_j^*\|^2 \int_{\Delta_n} \left| \sum_{k=1}^m p_k(\operatorname{Re} \zeta, \operatorname{Im} \zeta) \right|^2 d(E(\zeta)x, x) \\ &\geq \frac{1}{n} \|x\|^2 + \|R_j R_j^*\|^2 \left(\sum_{k=1}^m p_k(\Theta(A), \Theta(B)) \right) x \|^2. \end{aligned}$$

By (3.4) this inequality can only hold for $x = 0$. Since Δ_n is open, by the Spectral Theorem for normal operators on Hilbert spaces we have $\Delta_n \subseteq \rho(N)$. The asserted inclusion now follows from

$$\{z \in \mathbb{C} : |p_j(\operatorname{Re} z, \operatorname{Im} z)| > \|R_j R_j^*\| \cdot \left| \sum_{k=1}^m p_k(\operatorname{Re} z, \operatorname{Im} z) \right|\} = \bigcup_{n \in \mathbb{N}} \Delta_n.$$

□

In the following let I the ideal $\langle p_1, \dots, p_m \rangle$ generated by the real definitizing polynomials p_1, \dots, p_m in the ring $\mathbb{C}[x, y]$. The variety $V(I)$ is the set of all common zeros $a = (a_1, a_2) \in \mathbb{C}^2$ of all $p \in I$. Clearly, $V(I)$ coincides with the set of all $a \in \mathbb{C}^2$ such that $p_1(a_1, a_2) = \dots = p_m(a_1, a_2) = 0$. $V_{\mathbb{R}}(I)$ is the set of all $a \in \mathbb{R}^2$, which belong to $V(I)$. It is convenient for our purposes, to consider $V_{\mathbb{R}}(I)$ as a subset of \mathbb{C} :

$$\begin{aligned} V_{\mathbb{R}}(I) &:= \{z \in \mathbb{C} : f(\operatorname{Re} z, \operatorname{Im} z) = 0 \text{ for all } f \in I\} \\ &= \{z \in \mathbb{C} : p_k(\operatorname{Re} z, \operatorname{Im} z) = 0 \text{ for all } k \in \{1, \dots, m\}\}. \end{aligned} \quad (3.6)$$

3.6 Corollary. *Let E denote the special measure of $\Theta(N)$. Then we have*

$$R_j R_j^* E(\mathbb{C} \setminus V_{\mathbb{R}}(I)) = E(\mathbb{C} \setminus V_{\mathbb{R}}(I)) R_j R_j^* = \int_{\mathbb{C} \setminus V_{\mathbb{R}}(I)} \frac{p_j(\operatorname{Re} z, \operatorname{Im} z)}{\sum_{k=1}^m p_k(\operatorname{Re} z, \operatorname{Im} z)} dE(z).$$

Proof. First note that the integral on the right hand side exists as a bounded operator, because by Lemma 3.5 we have $|p_j(\operatorname{Re} z, \operatorname{Im} z)| \leq \|R_j R_j^*\| \cdot \left| \sum_{k=1}^m p_k(\operatorname{Re} z, \operatorname{Im} z) \right|$ for $z \in \sigma(\Theta(N))$. The first equality is known from Corollary 2.2.

Concerning the second equality, note that both sides vanish on the range of $E(V_{\mathbb{R}}(I))$. Its orthogonal complement $\mathcal{Q} := \operatorname{ran} E(\mathbb{C} \setminus V_{\mathbb{R}}(I))$ is invariant under

$$\int \left(\sum_{k=1}^m p_k(\operatorname{Re} z, \operatorname{Im} z) \right) dE(z) = \sum_{k=1}^m p_k(\Theta(A), \Theta(B)).$$

By Lemma 3.5 the restriction of this operator to \mathcal{Q} is injective, and hence, has dense range in \mathcal{Q} . If x belongs to this dense range, i.e. $x = (\sum_{k=1}^m p_k(\Theta(A), \Theta(B)))y$ with $y \in \mathcal{Q}$, then

$$\begin{aligned} \int_{\mathbb{C} \setminus V_{\mathbb{R}}(I)} \frac{p_j(\operatorname{Re} z, \operatorname{Im} z)}{\sum_{k=1}^m p_k(\operatorname{Re} z, \operatorname{Im} z)} dE(z)x &= \int_{\mathbb{C} \setminus V_{\mathbb{R}}(I)} p_j(\operatorname{Re} z, \operatorname{Im} z) dE(z)y \\ &= p_j(\Theta(A), \Theta(B))y = R_j R_j^* \left(\sum_{k=1}^m p_k(\Theta(A), \Theta(B)) \right) y = R_j R_j^* x. \end{aligned}$$

By a density argument the second asserted equality of the present corollary holds true on \mathcal{Q} and in turn on \mathcal{H} . \square

3.7 Remark. In Proposition 3.3 the case that $p_j(A, B) = 0$ for some j , or even for all j , is not excluded, and yields $\mathcal{H}_j = \{0\}$, $T_j = 0$ and $R_j = 0$ (in Lemma 3.4), or even $\mathcal{H} = \{0\}$ and $T = 0$. Also the remaining results hold true, if we interpret $\rho(R)$ as \mathbb{C} and $\sigma(R)$ as \emptyset for the only possible linear operator $R = (0 \mapsto 0)$ on the vector space $\{0\}$. \diamond

4 An Abstract Functional Calculus

In this section let \mathcal{K} be again a Krein space, $N \in B(\mathcal{K})$ be a definitizable normal operator. Let I be the ideal in $\mathbb{C}[x, y]$, which is generated by all real definitizing polynomials. By the ascending chain condition for the ring $\mathbb{C}[x, y]$ (see for example [CLO1], Theorem 7, Chapter 2, §5) I is generated by finitely many real definitizing polynomials p_1, \dots, p_m , i.e. $I = \langle p_1, \dots, p_m \rangle$. We employ the same notion as in the previous sections for these polynomials p_1, \dots, p_m . In particular, E_j (E) denotes the spectral measure of $\Theta_j(N)$ on \mathcal{H}_j ($\Theta(N)$ on \mathcal{H}).

We also make the convention that for $p \in \mathbb{C}[x, y]$ and $z \in \mathbb{C}$ we write $p(z)$ short for $p(\operatorname{Re} z, \operatorname{Im} z)$.

4.1 Lemma. *For a bounded and measurable $f : \sigma(\Theta(N)) \rightarrow \mathbb{C}$ and $j \in \{1, \dots, m\}$ we have*

$$\begin{aligned} \Xi_j \left(\int_{\sigma(\Theta_j(N))} f dE_j \right) &= \\ &\Xi \left(\int_{\sigma(\Theta(N)) \setminus V_{\mathbb{R}}(I)} f \frac{p_j}{\sum_{l=1}^m p_l} dE + R_j R_j^* \int_{\sigma(\Theta(N)) \cap V_{\mathbb{R}}(I)} f dE \right). \end{aligned}$$

Proof. By (2.5) the left hand side coincides with

$$\Xi \left(R_j R_j^* \int_{\sigma(\Theta(N)) \setminus V_{\mathbb{R}}(I)} f dE + R_j R_j^* \int_{\sigma(\Theta(N)) \cap V_{\mathbb{R}}(I)} f dE \right).$$

$\int_{\sigma(\Theta(N)) \setminus V_{\mathbb{R}}(I)} f dE = E(\mathbb{C} \setminus V_{\mathbb{R}}(I)) \int_{\sigma(\Theta(N)) \setminus V_{\mathbb{R}}(I)} f dE$ together with Corollary 3.6 prove the equality. \square

4.2 Lemma. Let $f, g : \sigma(\Theta(N)) \rightarrow \mathbb{C}$ be bounded and measurable, and let $r \in \mathbb{C}[x, y]$. For $j, k \in \{1, \dots, m\}$ we then have

$$\begin{aligned} r(A, B) \Xi_j \left(\int_{\sigma(\Theta_j(N))} f dE_j \right) &= \Xi_j \left(\int_{\sigma(\Theta_j(N))} f dE_j \right) r(A, B) \\ &= \Xi_j \left(\int_{\sigma(\Theta_j(N))} r f dE_j \right), \end{aligned} \quad (4.1)$$

and

$$\begin{aligned} \Xi_j \left(\int_{\sigma(\Theta_j(N))} f dE_j \right) \Xi_k \left(\int_{\sigma(\Theta_k(N))} g dE_k \right) \\ = \Xi \left(\int_{\sigma(\Theta(N))} f g \frac{p_j p_k}{\sum_{l=1}^m p_l} dE \right) \\ = \Xi_j \left(\int_{\sigma(\Theta_j(N))} f g p_k dE_j \right) = \Xi_k \left(\int_{\sigma(\Theta_k(N))} f g p_j dE_k \right). \end{aligned} \quad (4.2)$$

Proof. By Lemma 5.11 in [KP] we have

$$r(A, B) \Xi_j(D) = \Xi_j(\Theta(r(A, B))D) = \Xi_j(r(\Theta_j(A), \Theta_j(B))D),$$

$$\Xi_j(D)r(A, B) = \Xi_j(D\Theta_j(r(A, B))) = \Xi_j(Dr(\Theta_j(A), \Theta_j(B)))$$

for $D \in (T^+T)'$. For $D = \int_{\sigma(\Theta_j(N))} f dE_j$ this implies (4.1).

According to (2.5) the expression in (4.2) coincides with

$$\Xi \left(R_j R_j^* \int_{\sigma(\Theta(N))} f dE \right) \Xi \left(R_k R_k^* \int_{\sigma(\Theta(N))} g dE \right).$$

By Lemma 5.11 and Theorem 5.8 in [KP], we also know that $\Xi(D_1)\Xi(D_2) = \Xi(T^+T D_1 D_2) = \Xi(\Theta(TT^+)D_1 D_2)$, where (see Proposition 3.3 and (3.6))

$$\Theta(TT^+) = \sum_{l=1}^m p_l(\Theta(A), \Theta(B)) = \int \sum_{l=1}^m p_l dE = \left(\int \sum_{l=1}^m p_l dE \right) E(\mathbb{C} \setminus V_{\mathbb{R}}(I)).$$

Therefore, by Corollary 3.6 and the fact, that $E(\mathbb{C} \setminus V_{\mathbb{R}}(I))$ commutes with $\int_{\sigma(\Theta(N))} f dE$, (4.2) can be written as

$$\begin{aligned} \Xi \left(\left(\int \sum_{l=1}^m p_l dE \right) \left(\int \frac{p_j}{\sum_{l=1}^m p_l} dE \right) \left(\int f dE \right) \left(\int \frac{p_k}{\sum_{l=1}^m p_l} dE \right) \left(\int g dE \right) \right) = \\ \Xi \left(\int_{\sigma(\Theta(N))} f g \frac{p_j p_k}{\sum_{l=1}^m p_l} dE \right). \end{aligned}$$

The remaining equalities follow from Lemma 4.1 since the respective integrands vanish on $V_{\mathbb{R}}(I)$. \square

4.3 Lemma. For a bounded and measurable $f : \sigma(\Theta(N)) \rightarrow \mathbb{C}$ and $j \in \{1, \dots, m\}$ the operator $\Xi_j \left(\int_{\sigma(\Theta_j(N))} f dE_j \right)$ belongs to $\{N, N^+\}''$.

Proof. Take $C \in \{N, N^+\}' = \{A, B\}' \subseteq \bigcap_{j=1, \dots, m} (T_j T_j^+)'$; see (3.3). From Lemma 5.11 in [KP] we conclude

$$C \Xi_j \left(\int_{\sigma(\Theta_j(N))} f dE_j \right) = \Xi_j \left(\Theta_j(C) \left(\int_{\sigma(\Theta_j(N))} f dE_j \right) \right).$$

Since Θ_j is a homomorphism, $\Theta_j(C)$ commutes with $\Theta_j(N)$ and, in turn, with $\int_{\sigma(\Theta_j(N))} f dE_j$. Hence, employing Lemma 5.11 in [KP] once more, the above expression coincides with

$$\Xi_j \left(\left(\int_{\sigma(\Theta_j(N))} f dE_j \right) \Theta_j(C) \right) = \Xi_j \left(\int_{\sigma(\Theta_j(N))} f dE_j \right) C.$$

□

4.4 Definition. Denoting by $\mathfrak{B}(\sigma(\Theta(N)))$ the $*$ -algebra of complex valued, bounded and measurable functions on $\sigma(\Theta(N))$, for $(r, f_1, \dots, f_m) \in \mathcal{R} := \mathbb{C}[x, y] \times \mathfrak{B}(\sigma(\Theta(N))) \times \dots \times \mathfrak{B}(\sigma(\Theta(N)))$ we set

$$\Psi(r, f_1, \dots, f_m) := r(A, B) + \sum_{k=1}^m \Xi_k \left(\int_{\sigma(\Theta_k(N))} f_k dE_k \right).$$

By \mathcal{N} we denote the set of all $(r, f_1, \dots, f_m) \in \mathcal{R}$ such that

$$r + \sum_{k=1}^m f_k p_k = 0 \quad \text{on} \quad \sigma(\Theta(N)) \setminus V_{\mathbb{R}}(I)$$

and such that there exist $u_1, \dots, u_m \in \mathbb{C}[x, y]$ with $r = \sum_{k=1}^m u_k p_k$ and

$$(f_j + u_j)(z) = 0 \quad \text{for} \quad j = 1, \dots, m, \quad z \in V_{\mathbb{R}}(I) \cap \sigma(\Theta(N)).$$

◇

4.5 Remark. Obviously, Ψ is linear. From $\Xi_j(D^*) = \Xi_j(D)^+$ we easily deduce $\Psi(r^\#, \overline{f_1}, \dots, \overline{f_m}) = \Psi(r, f_1, \dots, f_m)^*$. Moreover, \mathcal{N} constitutes a linear subspace of \mathcal{R} invariant under $\cdot^\# : (r, f_1, \dots, f_m) \mapsto (r^\#, \overline{f_1}, \dots, \overline{f_m})$. ◇

4.6 Lemma. If $(r, f_1, \dots, f_m) \in \mathcal{N}$, then $\Psi(r, f_1, \dots, f_m) = 0$.

Proof. Due to (3.5) $r = \sum_{k=1}^m u_k p_k$ implies

$$r(A, B) = \sum_{k=1}^m p_k(A, B) u_k(A, B) = \sum_{k=1}^m \Xi_k(u_k(\Theta_k(A), \Theta_k(B))).$$

From this and Lemma 4.1 we obtain

$$\begin{aligned} \Psi(r, f_1, \dots, f_m) &= \sum_{k=1}^m \Xi_k \left(\int_{\sigma(\Theta_k(N))} (f_k + u_k) dE_k \right) = \\ &= \Xi \left(\int_{\sigma(\Theta(N)) \setminus V_{\mathbb{R}}(I)} \sum_{k=1}^m \frac{f_k p_k + u_k p_k}{\sum_{l=1}^m p_l} dE + \sum_{k=1}^m R_k R_k^* \int_{\sigma(\Theta(N)) \cap V_{\mathbb{R}}(I)} (f_k + u_k) dE \right) = 0. \end{aligned}$$

□

4.7 Lemma. For $(r, f_1, \dots, f_m), (s, g_1, \dots, g_m) \in \mathcal{R}$ have

$$\begin{aligned} \Psi(r, f_1, \dots, f_m) \Psi(s, g_1, \dots, g_m) &= \\ &= \Psi(rs, rg_1 + sf_1 + f_1 \sum_{k=1}^m g_k p_k, \dots, rg_m + sf_m + f_m \sum_{k=1}^m g_k p_k) \\ &= \Psi(rs, rg_1 + sf_1 + g_1 \sum_{k=1}^m f_k p_k, \dots, rg_m + sf_m + g_m \sum_{k=1}^m f_k p_k). \end{aligned}$$

Proof. By Lemma 4.2 we have

$$\begin{aligned} \Psi(r, f_1, \dots, f_m) \Psi(s, g_1, \dots, g_m) &= r(A, B) s(A, B) \\ &+ \sum_{k=1}^m r(A, B) \Xi_k \left(\int_{\sigma(\Theta_k(N))} g_k dE_k \right) + \sum_{j=1}^m \Xi_j \left(\int_{\sigma(\Theta_j(N))} f_j dE_j \right) s(A, B) \\ &+ \sum_{j,k=1}^m \Xi_j \left(\int_{\sigma(\Theta_j(N))} f_j dE_j \right) \Xi_k \left(\int_{\sigma(\Theta_k(N))} g_k dE_k \right) \\ &= (rs)(A, B) + \sum_{k=1}^m \Xi_k \left(\int_{\sigma(\Theta_k(N))} rg_k dE_k \right) + \sum_{j=1}^m \Xi_j \left(\int_{\sigma(\Theta_j(N))} sf_j dE_j \right) \\ &+ \sum_{j=1}^m \Xi_j \left(\sum_{k=1}^m \int_{\sigma(\Theta_j(N))} f_j g_k p_k dE_j \right), \end{aligned}$$

where this last term can also be written as

$$\sum_{j=1}^m \Xi_j \left(\sum_{k=1}^m \int_{\sigma(\Theta_j(N))} f_k g_j p_k dE_j \right).$$

□

4.8 Definition. We provide \mathcal{R} with a multiplication in the following way:

$$\begin{aligned} (r, f_1, \dots, f_m) \cdot (s, g_1, \dots, g_m) &:= \\ &= (rs, rg_1 + sf_1 + f_1 \sum_{j=1}^m g_j p_j, \dots, rg_m + sf_m + f_m \sum_{j=1}^m g_j p_j). \end{aligned}$$

◇

4.9 Remark. Obviously, \cdot is bilinear and compatible with $\cdot^\#$ as defined in Remark 4.5. It is elementary to check its associativity.

Moreover, for $(r, f_1, \dots, f_m) \in \mathcal{N}$ and $(s, g_1, \dots, g_m) \in \mathcal{R}$ we have $rs + \sum_{j=1}^m p_j (rg_j + sf_j + f_j \sum_{k=1}^m g_k p_k) = (r + \sum_{j=1}^m f_j p_j)(s + \sum_{k=1}^m g_k p_k) = 0$ on $\mathbb{C} \setminus V_{\mathbb{R}}(I)$. For the corresponding $u_1, \dots, u_m \in \mathbb{C}[x, y]$ with $r = \sum_{j=1}^m u_j p_j$ and $(f_j + u_j)(z) = 0$ for all $z \in V_{\mathbb{R}}(I)$ we have $rs = \sum_{j=1}^m (u_j s) p_j$ and

$$rg_j + sf_j + f_j \sum_{k=1}^m g_k p_k + u_j s = rg_j + f_j \sum_{k=1}^m g_k p_k = 0$$

on $V_{\mathbb{R}}(I)$ since r and the p_j vanish there. Hence, \mathcal{N} is a right ideal. Similarly, one shows that it is also a left ideal. Finally, the commutator

$$(r, f_1, \dots, f_m) \cdot (s, g_1, \dots, g_m) - (s, g_1, \dots, g_m) \cdot (r, f_1, \dots, f_m) = \\ (0, \sum_{j=1}^m (f_1 g_j - g_1 f_j) p_j, \dots, \sum_{j=1}^m (f_m g_j - g_m f_j) p_j)$$

belongs to \mathcal{N} . Consequently, \mathcal{R}/\mathcal{N} is a commutative $*$ -algebra. \diamond

Gathering the previous results we obtain the final result of the present section.

4.10 Theorem. $\Psi/\mathcal{N} : (r, f_1, \dots, f_m) + \mathcal{N} \mapsto \Psi(r, f_1, \dots, f_m)$ is a well-defined $*$ -homomorphism from \mathcal{R}/\mathcal{N} into $\{N, N^+\}'' \subseteq B(\mathcal{K})$.

5 Algebra of Zero-dimensional Ideals

By the Noether-Lasker Theorem (see for example [CLO1], Theorem 7, Chapter 4, §7) any ideal I in $\mathbb{C}[x, y]$ admits a minimal primary decomposition

$$I = Q_1 \cap \dots \cap Q_l. \quad (5.1)$$

Q_j being a primary ideal means that $fg \in Q_j$ implies $f \in Q_j$ or $g^k \in Q_j$ for some $k \in \mathbb{N}$, and minimal means that $Q_j \not\supseteq \bigcap_{i \neq j} Q_i$ for all $j = 1, \dots, l$ and $P_j \neq P_i$ for $i \neq j$, where P_j denotes the radical

$$\sqrt{Q_j} := \{f \in \mathbb{C}[x, y] : f^k \in Q_j \text{ for some } k \in \mathbb{N}\}.$$

For an ideal I in $\mathbb{C}[x, y]$ such a decomposition is in general not unique. Nevertheless, the First Uniqueness Theorem on minimal primary decompositions states that the number $l \in \mathbb{N}$ and the radicals P_1, \dots, P_l are uniquely determined by I ; see for example [BW], Theorem 8.55 on page 362. Moreover, the Second Uniqueness Theorem on minimal primary decompositions says that if $Q'_1 \cap \dots \cap Q'_l = I = Q_1 \cap \dots \cap Q_l$ are minimal primary decompositions ordered such that $P_j = \sqrt{Q_j} = \sqrt{Q'_j}$ for $j = 1, \dots, l$ and if P_k is minimal in $\{P_1, \dots, P_l\}$ with respect to \subseteq , then $Q'_k = Q_k$; see for example [BW], Theorem 8.56 on page 364.

Assume now that I is a zero-dimensional ideal in $\mathbb{C}[x, y]$, i.e.

$$\dim \mathbb{C}[x, y]/I < \infty.$$

For necessary and sufficient conditions see for example [BW], Theorem 6.54 and Corollary 6.56 on pages 274 and 275 and [CLO2], page 39 and 40. Let (5.1) be a minimal primary decomposition. Then any Q_j , and in turn $P_j \supseteq Q_j$, is also zero-dimensional. In particular, $\mathbb{C}[x, y]/P_j$ is a finite integral domain, and hence, a field. In turn, the radicals P_1, \dots, P_l of Q_1, \dots, Q_l are maximal ideals. By [CLO1], Theorem 11, Chapter 4, §5, this means that the P_j are generated by $x - a_{x,j}, y - a_{y,j}$, i.e. $P_j = \langle x - a_{x,j}, y - a_{y,j} \rangle$, for pairwise distinct $a_j = (a_{x,j}, a_{y,j}) \in \mathbb{C}^2$. Consequently, any P_k is minimal in $\{P_1, \dots, P_l\}$, and by what was said above, (5.1) is the unique minimal primary decomposition of I .

By Hilbert's Nullstellensatz (see for example [CLO1], Theorem 2, Chapter 4, §1) the set $V(Q_j)$ of common zeros in \mathbb{C}^2 of all $f \in Q_j$ coincides with $V(P_j) = \{a_j\}$. By [CLO1], Theorem 7, Chapter 4, §3, we also have

$$V(I) = V(Q_1) \cup \cdots \cup V(Q_l) = \{a_1, \dots, a_l\}.$$

Since $V(Q_j + Q_i) = V(Q_j) \cap V(Q_i) = \{a_j\} \cap \{a_i\} = \emptyset$ (see [CLO1], Theorem 4, Chapter 4, §3) for $i \neq j$, the weak Nullstellensatz (see for example [CLO1], Theorem 1, Chapter 4, §1) yields $Q_j + Q_i = \mathbb{C}[x, y]$. Hence, by the Chinese Remainder Theorem the mapping

$$\phi: \begin{cases} \mathbb{C}[x, y]/I & \rightarrow (\mathbb{C}[x, y]/Q_1) \times \cdots \times (\mathbb{C}[x, y]/Q_l), \\ x + I & \mapsto (x + Q_1, \dots, x + Q_l) \end{cases} \quad (5.2)$$

constitutes an isomorphism, and $I = \prod_{j=1}^l Q_j$.

5.1 Remark.

1. Since the ring $\mathbb{C}[x, y]/Q_j$ is finite dimensional, its invertible elements $f + Q_j$ are exactly those, for which $fg \in Q_j$ implies $g \in Q_j$. Q_j being primary this is equivalent to $f \notin P_j$. Hence, $f + Q_j$ is invertible in $\mathbb{C}[x, y]/Q_j$ if and only if $f(a_j) \neq 0$.
2. As $\sqrt{Q_j} = P_j$ we have $(x - a_{x,j})^m, (y - a_{y,j})^n \in Q_j$ for sufficiently large $m, n \in \mathbb{N}$. Therefore, the ideal $P_j \cdot Q_j$ contains $(x - a_{x,j})^{m+1}, (y - a_{y,j})^{n+1}$. Thus, $P_j \cdot Q_j$ is also zero-dimensional and $\sqrt{P_j \cdot Q_j} = P_j$.

◇

5.2 Definition. For $a \in V(I)$ we set by $Q(a) := Q_j$ and $P(a) := P_j$, where j is such that $a = a_j$. By $d_x(a)$ ($d_y(a)$) we denote the smallest natural number m (n) such that $(x - a_x)^m \in Q(a)$ ($(y - a_y)^n \in Q(a)$). Moreover, for $a \in V(I)$ we set

$$\mathcal{A}(a) := \mathbb{C}[x, y]/(P(a) \cdot Q(a)) \quad \text{and} \quad \mathcal{B}(a) := \mathbb{C}[x, y]/Q(a).$$

◇

Since $P(a) \cdot Q(a)$ and $Q(a)$ are ideals with finite codimension satisfying $P(a) \cdot Q(a) \subseteq Q(a)$, $\mathcal{A}(a)$ and $\mathcal{B}(a)$ are finite dimensional algebras with $\dim \mathcal{A}(a) \geq \dim \mathcal{B}(a)$.

5.3 Remark. Assume that I is invariant under $\#$, where $f^\#(x, y) := \overline{f(\bar{x}, \bar{y})}$. This is for sure the case if I is generated by real polynomial p_1, \dots, p_m . Then $V(I) \subseteq \mathbb{C}^2$ is invariant under $(z, w) \mapsto (z, w)^\# := (\bar{z}, \bar{w})$. Moreover, it is elementary to check that with Q also $Q^\#$ is a primary ideal. Hence, with $I = Q_1 \cap \cdots \cap Q_l$ also $I = I^\# = Q_1^\# \cap \cdots \cap Q_l^\#$ is a minimal primary decomposition. By the uniqueness of the minimal primary decomposition for our zero dimensional ideal I one has $Q(a)^\# = Q(a^\#)$ for all $a \in V(I)$.

Consequently, $f \mapsto f^\#$ induces a conjugate linear bijection from $\mathcal{A}(a)$ ($\mathcal{B}(a)$) onto $\mathcal{A}(a^\#)$ ($\mathcal{B}(a^\#)$). ◇

For the following note that if we conversely start with primary and zero-dimensional ideals Q_1, \dots, Q_l with $\sqrt{Q_i} \neq \sqrt{Q_j}$ for $i \neq j$, then $I := Q_1 \cap \cdots \cap Q_l$ is also zero-dimensional, and by the above mentioned uniqueness statement, $Q_1 \cap \cdots \cap Q_l$ is indeed the unique minimal primary decomposition of I .

5.4 Proposition. *Let I be a zero-dimensional ideal in $\mathbb{C}[x, y]$ which is generated by p_1, \dots, p_m , and let $I = \bigcap_{a \in V(I)} Q(a)$ be its unique primary decomposition. Assume that W is a subset of $V(I)$. Then*

$$J := \bigcap_{a \in V(I) \setminus W} Q(a) \cap \bigcap_{a \in W} (P(a) \cdot Q(a))$$

is also a zero-dimensional ideal satisfying $J \subseteq I$. The mapping

$$\psi : \begin{cases} \mathbb{C}[x, y]/J & \rightarrow \bigtimes_{a \in V(I) \setminus W} (\mathbb{C}[x, y]/Q(a)) \times \bigtimes_{a \in W} (\mathbb{C}[x, y]/(P(a) \cdot Q(a))), \\ x + I & \mapsto ((x + Q(a))_{a \in V(I) \setminus W}, (x + (P(a) \cdot Q(a)))_{a \in W}) \end{cases}$$

is an isomorphism, and any $p \in J$ can be written in the form $p = \sum_j u_j p_j$, where $u_j(a) = 0$ for all $a \in W$.

Proof. We already mentioned that $P(a) \cdot Q(a)$ is zero-dimensional with $\sqrt{P(a) \cdot Q(a)} = P(a)$ and that the intersection $J = \bigcap_{a \in V(I) \setminus W} Q(a) \cap \bigcap_{a \in W} P(a) \cdot Q(a)$ is the unique primary decomposition of the zero-dimensional J . The isomorphism property of ψ is a special case of the corresponding fact concerning ϕ ; see (5.2). We also have

$$\begin{aligned} J &= \prod_{a \in V(I) \setminus W} Q(a) \cdot \prod_{a \in W} P(a) \cdot Q(a) = \prod_{a \in V(I)} Q(a) \cdot \prod_{a \in W} P(a) \\ &= I \cdot \prod_{a \in W} P(a) = \left\langle p_1 \cdot \prod_{a \in W} P(a), \dots, p_m \cdot \prod_{a \in W} P(a) \right\rangle. \end{aligned}$$

This means that any $p \in J$ has a representation $p = \sum_j u_j p_j$ with $u_j \in \prod_{a \in W} P(a) = \bigcap_{a \in W} P(a)$. Hence, $u_j(a) = 0$ for all $a \in W$. \square

5.5 Example. Assume that I is generated by two polynomial $p_1, p_2 \in \mathbb{C}[x, y]$ such that p_1 only depend on x and p_2 only depends on y . The set $V(I)$ of common zeros of I , or equivalently of p_1 and p_2 , in \mathbb{C}^2 then consists of all points of the form (z, w) , where $z \in \mathbb{C}$ is a zero of p_1 and $w \in \mathbb{C}$ is a zero of p_2 , i.e. $V(I) = p_1^{-1}\{0\} \times p_2^{-1}\{0\}$. For $z \in p_1^{-1}\{0\}$ denote by $\mathfrak{d}_1(z)$ p_1 's degrees of the zero at z , and for $w \in p_2^{-1}\{0\}$ denote by $\mathfrak{d}_2(w)$ p_2 's degrees of the zero at w .

Given $p(x, y) \in \mathbb{C}[x, y]$ we can apply polynomial division in one variable twice, once with respect to x and once y , in order to see that

$$p(x, y) = p_1(x) \cdot u(x, y) + p_2(y) \cdot v(x, y) + q(x, y)$$

with $u(x, y), v(x, y), q(x, y) \in \mathbb{C}[x, y]$ such that the degree of $q(x, y)$, seen as a polynomial on x , is less than the degree of p_1 , and such that the degree of $q(x, y)$, seen as a polynomial on y , is less than the degree of p_2 ; see Lemma 4.8 in [K]. Hence, I is zero-dimensional. Moreover, writing $p_1(x)$ and $p_2(y)$ as products of linear factors, it follows that $p \in I$ if and only if

$$p \in \langle (x - z)^{\mathfrak{d}_1(z)}, (y - w)^{\mathfrak{d}_2(w)} \rangle := Q((z, w)), \quad (5.3)$$

for all $z \in p_1^{-1}\{0\}, w \in p_2^{-1}\{0\}$. Since $Q((z, w))$ is a primary ideal in $\mathbb{C}[x, y]$,

$$I = \bigcap_{(z, w) \in p_1^{-1}\{0\} \times p_2^{-1}\{0\}} Q((z, w))$$

is the minimal primary decomposition of I . For the respective radicals we have $P((z, w)) = \langle x - z, y - w \rangle$. Moreover, $P((z, w)) \cdot Q((z, w))$ coincides with

$$\langle (x - z)^{\mathfrak{d}_1(z)+1}, (x - z)^{\mathfrak{d}_1(z)}(y - w), (x - z)(y - w)^{\mathfrak{d}_2(w)}, (y - w)^{\mathfrak{d}_2(w)+1} \rangle.$$

Therefore, $\mathcal{A}((z, w)) = \mathbb{C}[x, y]/(P((z, w)) \cdot Q((z, w)))$ ($\mathcal{B}((z, w)) = \mathbb{C}[x, y]/Q((z, w))$) is isomorphic to $\mathcal{A}_{\mathfrak{d}_1(z), \mathfrak{d}_2(w)}$ ($\mathcal{B}_{\mathfrak{d}_1(z), \mathfrak{d}_2(w)}$) as introduced in Definition 4.1, [K]. \diamond

6 Function classes

In the present section we make the same assumptions and use the same notation as in Section 4. In addition, we assume that the ideal I generated by all real definitizing polynomials is zero-dimensional. We fix real, definitizing polynomials p_1, \dots, p_m which generate I . For the zero-dimensional I we apply the same notation as in the previous section.

The variety $V(I) = \{a_1, \dots, a_l\} \subseteq \mathbb{C}^2$ of common zeros of all $f \in I$ will be split up as

$$V(I) = \underbrace{(V(I) \cap \mathbb{R}^2)}_{=V_{\mathbb{R}}(I)} \dot{\cup} (V(I) \setminus \mathbb{R}^2),$$

where we consider $V_{\mathbb{R}}(I)$ as a subset of \mathbb{C} ; see (3.6).

6.1 Definition. By \mathcal{M}_N we denote the set of functions ϕ defined on

$$\underbrace{(\sigma(\Theta(N)) \cup V_{\mathbb{R}}(I))}_{\subseteq \mathbb{C}} \dot{\cup} \underbrace{(V(I) \setminus \mathbb{R}^2)}_{\subseteq \mathbb{C}^2}$$

with $\phi(z) \in \mathbb{C}$ for $z \in \sigma(\Theta(N)) \setminus V_{\mathbb{R}}(I)$, $\phi(z) \in \mathcal{A}(z)$ for $z \in V_{\mathbb{R}}(I)$, $\phi(z) \in \mathcal{B}(z)$ for $z \in V(I) \setminus \mathbb{R}^2$.

We provide \mathcal{M}_N pointwise with scalar multiplication, addition and multiplication. We also define a conjugate linear involution $\#$ on \mathcal{M}_N by

$$\begin{aligned} \phi^\#(z) &:= \overline{\phi(z)} \quad \text{for } z \in \sigma(\Theta(N)) \setminus V_{\mathbb{R}}(I), \\ \phi^\#(z) &:= \phi(z)^\# \quad \text{for } z \in V_{\mathbb{R}}(I) \\ \phi^\#(\xi, \eta) &:= \phi(\bar{\xi}, \bar{\eta})^\# \quad \text{for } (\xi, \eta) \in V(I) \setminus \mathbb{R}^2. \end{aligned}$$

\diamond

With the operations introduced above \mathcal{M}_N is a commutative $*$ -algebra as can be verified in a straight forward manner; see Remark 5.3.

6.2 Definition. Let $f : \text{dom } f \rightarrow \mathbb{C}$ be a function with $\text{dom } f \subseteq \mathbb{C}^2$ such that $\tau(\sigma(\Theta(N)) \cup V_{\mathbb{R}}(I)) \subseteq \text{dom } f$, where $\tau : \mathbb{C} \rightarrow \mathbb{C}^2$, $(x + iy) \mapsto (x, y)$, such that $f \circ \tau$ is sufficiently smooth – more exactly, at least $d_x(z) + d_y(z) - 1$ times continuously differentiable – on a sufficiently small open neighbourhood z for each $z \in V_{\mathbb{R}}(I)$, and such that f is holomorphic on an open neighbourhood of $V(I) \setminus \mathbb{R}^2$ ($\subseteq \mathbb{C}^2$).

Then f can be considered as an element f_N of \mathcal{M}_N by setting $f_N(z) := f \circ \tau(z)$ for $z \in \sigma(\Theta(N)) \setminus V_{\mathbb{R}}(I)$, by

$$f_N(z) := \sum_{(k,l) \in J(z)} \frac{1}{k!l!} \frac{\partial^{k+l}}{\partial a^k \partial b^l} f \circ \tau(a+ib)|_{a+ib=z} \cdot (x - \operatorname{Re} z)^k (y - \operatorname{Im} z)^l + (P(z) \cdot Q(z)) \in \mathcal{A}(z)$$

for $z \in V_{\mathbb{R}}(I)$, where

$$J(z) = (\{0, \dots, d_x(z) - 1\} \times \{0, \dots, d_y(z) - 1\}) \cup \{(d_x(z), 0), (0, d_y(z))\},$$

and by

$$f_N(\xi, \eta) := \sum_{k=0}^{d_x(\xi, \eta)-1} \sum_{l=0}^{d_y(\xi, \eta)-1} \frac{1}{k!l!} \frac{\partial^{k+l}}{\partial z^k \partial w^l} f(z, w)|_{(z,w)=(\xi, \eta)} \cdot (x - \xi)^k (y - \eta)^l + Q((\xi, \eta)) \in \mathcal{B}((\xi, \eta)),$$

for $(\xi, \eta) \in V(I) \setminus \mathbb{R}^2$. \diamond

6.3 Remark. By the Leibniz rule $f \mapsto f_N$ is compatible with multiplication. Obviously, it is also compatible with addition and scalar multiplication. If we define for a function f as in Definition 6.2 the function $f^\#$ by $f^\#(z, w) = f(\bar{z}, \bar{w})$, $(z, w) \in \operatorname{dom} f$, then we also have $(f^\#)_N = (f_N)^\#$. \diamond

6.4 Remark. A special type of functions f as in Definition 6.2 are polynomials in two variables, i.e. $f \in \mathbb{C}[x, y]$. Since for $z \in V_{\mathbb{R}}(I)$ and $(k, l) \notin J(z)$ we have $(x - \operatorname{Re} z)^k (y - \operatorname{Im} z)^l \in P(z) \cdot Q(z)$,

$$f_N(z) = f + (P(z) \cdot Q(z)) \in \mathcal{A}(z).$$

Similarly, $f_N(\xi, \eta) = f + Q((\xi, \eta)) \in \mathcal{B}((\xi, \eta))$ for $(\xi, \eta) \in V(I) \setminus \mathbb{R}^2$.

In particular, for $f = \mathbb{1}$ the element $f_N(z)$ is the multiplicative unite in $\mathcal{A}(z)$ or $\mathcal{B}(z)$ for all $z \in (\sigma(\Theta(N)) \cup V_{\mathbb{R}}(I)) \dot{\cup} (V(I) \setminus \mathbb{R}^2)$. \diamond

For the following recall for example from [CLO1], Theorem 4, Chapter 2, §5, that any ideal in $\mathbb{C}[x, y]$ always has a finite number of generators.

6.5 Definition. For any $w \in \sigma(\Theta(N)) \cap V_{\mathbb{R}}(I)$ such that w is not isolated in $\sigma(\Theta(N))$ let h_1, \dots, h_n be generators of the ideal $Q(w)$. For a sufficiently small neighbourhood $U(w)$ of w let $\chi_w : U(w) \setminus \{w\} \rightarrow [0, +\infty)$ be

$$\chi_w(z) := \max_{j=1, \dots, n} |h_j(z)|,$$

where $h_j(z)$, as usually, stands for $h_j(\operatorname{Re} z, \operatorname{Im} z)$. \diamond

Since w is a common zero of all $h \in Q(w)$, we have $\chi_w(z) \rightarrow 0$ for $z \rightarrow w$. Moreover, for any $h \in Q(w)$ the fact, that h_1, \dots, h_n are generators of $Q(w)$, yields $h = O(\chi_w)$ as $z \rightarrow w$.

Moreover, if χ'_w is defined in a similar manner starting with generators h'_1, \dots, h'_n , then $\chi'_w = O(\chi_w)$ and $\chi_w = O(\chi'_w)$ as $z \rightarrow w$. Hence, as far as it concerns the order of growth towards w , the expression χ_w does not depend on the actually chosen generators.

6.6 Definition. We denote by \mathcal{F}_N the set of all elements $\phi \in \mathcal{M}_N$ such that $z \mapsto \phi(z)$ is Borel measurable and bounded on $\sigma(\Theta(N)) \setminus V_{\mathbb{R}}(I)$, and such that for each $w \in \sigma(\Theta(N)) \cap V_{\mathbb{R}}(I)$, which is not isolated in $\sigma(\Theta(N))$,

$$\phi(z) - \phi(w)|_{x=\operatorname{Re} z, y=\operatorname{Im} z} = O(\chi_w(z)) \quad \text{as } \sigma(\Theta(N)) \setminus V_{\mathbb{R}}(I) \ni z \rightarrow w. \quad (6.1)$$

◇

Note that in (6.1) $\phi(w) \in \mathcal{A}(w)$ is a coset $p(x, y) + (P(w) \cdot Q(w))$ from $\mathbb{C}[x, y]/(P(w) \cdot Q(w))$, and $\phi(w)|_{x=\operatorname{Re} z, y=\operatorname{Im} z}$ stands for any representative of this coset $\phi(w)$ considered as a function of z . In (6.1) it does not matter what representative we take since $q = O(\chi_w)$ as $z \rightarrow w$ for any $q \in Q(w)$, and hence, for any $q \in (P(w) \cdot Q(w))$.

6.7 Remark. Assume that our zero-dimensional ideal I is generated by two definitizing polynomials $p_1 \in \mathbb{R}[x], p_2 \in \mathbb{R}[y]$ as in Example 5.5. For $w \in V_{\mathbb{R}}(I)$, i.e. $(\operatorname{Re} w, \operatorname{Im} w) \in V(I)$, we conclude from (5.3) in Example 5.5 that

$$\chi_w(z) := \max(|(\operatorname{Re} z - \operatorname{Re} w)^{\mathfrak{d}_1(\operatorname{Re} w)}|, |(\operatorname{Im} z - \operatorname{Im} w)^{\mathfrak{d}_2(\operatorname{Im} w)}|).$$

Therefore, in this case the function class \mathcal{F}_N here coincides exactly with the function class \mathcal{F}_N introduced in Definition 4.11, [K]. ◇

6.8 Example. For $(\xi, \eta) \in V(I) \setminus \mathbb{R}^2$ and $a \in \mathcal{B}((\xi, \eta))$ the function $a\delta_{(\xi, \eta)} \in \mathcal{M}_N$, which assumes the value a at (ξ, η) and the value zero on the rest of $(\sigma(\Theta(N)) \cup V_{\mathbb{R}}(I)) \setminus \{(\xi, \eta)\}$, trivially belongs to \mathcal{F}_N .

Correspondingly, $a\delta_w \in \mathcal{F}_N$ for a $w \in V_{\mathbb{R}}(I)$, which is an isolated point of $\sigma(\Theta(N)) \cup V_{\mathbb{R}}(I)$, and for $a \in \mathcal{A}(w)$. ◇

6.9 Remark. Let h be defined on an open subset D of \mathbb{R}^2 with values in \mathbb{C} . Moreover, assume that for given $m, n \in \mathbb{N}$ the function h is $m + n - 1$ times continuously differentiable. Finally, fix $w \in D$.

The well-known Taylor Approximation Theorem from multidimensional calculus then yields

$$h(z) = \sum_{j=0}^{m+n-2} \sum_{\substack{k, l \in \mathbb{N}_0 \\ k+l=j}} \frac{1}{k!l!} \frac{\partial^j h}{\partial x^k \partial y^l}(w) \operatorname{Re}(z-w)^k \operatorname{Im}(z-w)^l + O(|z-w|^{m+n-1})$$

for $z \rightarrow w$. Since

$$\begin{aligned} |z-w|^{m+n-1} &\leq 2^{m+n-1} \max(|\operatorname{Re}(z-w)|^{m+n-1}, |\operatorname{Im}(z-w)|^{m+n-1}) \\ &= O(\max(|\operatorname{Re}(z-w)|^m, |\operatorname{Im}(z-w)|^n)), \end{aligned}$$

and since $\operatorname{Re}(z-w)^k \operatorname{Im}(z-w)^l = O(\max(|\operatorname{Re}(z-w)|^m, |\operatorname{Im}(z-w)|^n))$ for $k \geq m$ or $l \geq n$, we also have

$$\begin{aligned} h(z) &= \sum_{k=0}^{m-1} \sum_{l=0}^{n-1} \frac{1}{k!l!} \frac{\partial^{k+l} h}{\partial x^k \partial y^l}(w) \operatorname{Re}(z-w)^k \operatorname{Im}(z-w)^l \\ &\quad + O(\max(|\operatorname{Re}(z-w)|^m, |\operatorname{Im}(z-w)|^n)). \end{aligned}$$

◇

6.10 Lemma. *Let $f : \text{dom } f (\subseteq \mathbb{C}^2) \rightarrow \mathbb{C}$ be a function with the properties mentioned in Definition 6.2. Then f_N belongs to \mathcal{F}_N .*

Proof. For a $w \in \sigma(\Theta(N)) \cap V_{\mathbb{R}}(I)$, which is not isolated in $\sigma(\Theta(N))$, and $z \in \sigma(\Theta(N)) \setminus V_{\mathbb{R}}(I)$ sufficiently near at w by Remark 6.9 the expression

$$f_N(z) - f_N(w)|_{x=\text{Re } z, y=\text{Im } z} =$$

$$f(\text{Re } z, \text{Im } z) - \sum_{(k,l) \in J(w)} \frac{1}{k!l!} \frac{\partial^{k+l} f}{\partial x^k \partial y^l}(\text{Re } w, \text{Im } w) \cdot (\text{Re } z - \text{Re } w)^k (\text{Im } z - \text{Im } w)^l$$

is a $O(\max(|\text{Re}(z - w)|^{d_x(w)}, |\text{Im}(z - w)|^{d_y(w)}))$, and therefore a $O(\chi_w(z))$ as $z \rightarrow w$. Consequently $f_N \in \mathcal{F}_N$. \square

6.11 Lemma. *If $\phi \in \mathcal{F}_N$ is such that $\phi(z)$ is invertible in $\mathbb{C}, \mathcal{A}(z)$ or $\mathcal{B}(z)$, respectively, for all $z \in (\sigma(\Theta(N)) \cup V_{\mathbb{R}}(I)) \dot{\cup} (V(I) \setminus \mathbb{R}^2)$ and such that $0 \in \mathbb{C}$ does not belong to the closure of $\phi(\sigma(\Theta(N)) \setminus V_{\mathbb{R}}(I))$, then $\phi^{-1} : z \mapsto \phi(z)^{-1}$ also belongs to \mathcal{F}_N .*

Proof. By the first assumption ϕ^{-1} is a well-defined object belonging to \mathcal{M}_N . Clearly, with ϕ also $z \mapsto \phi(z)^{-1} = \frac{1}{\phi(z)}$ is measurable on $\sigma(\Theta(N)) \setminus V_{\mathbb{R}}(I)$. By the second assumption of the present lemma $z \mapsto \phi(z)^{-1} = \frac{1}{\phi(z)}$ is bounded on this set.

It remains to verify (6.1) for ϕ^{-1} at each $w \in \sigma(\Theta(N)) \cap V_{\mathbb{R}}(I)$, which is not isolated in $\sigma(\Theta(N))$. To do so, first note that due to $\phi(w)$'s invertibility for $z \in \sigma(\Theta(N)) \setminus V_{\mathbb{R}}(I)$ sufficiently near at w we have $\phi(w)|_{x=\text{Re } z, y=\text{Im } z} = p(z) \neq 0$, where $p(x, y)$ is a representative of $\phi(w)$. Now calculate

$$\phi^{-1}(z) - \phi(w)^{-1}|_{x=\text{Re } z, y=\text{Im } z} = \tag{6.2}$$

$$= \frac{1}{\phi(z)} - \frac{1}{\phi(w)|_{x=\text{Re } z, y=\text{Im } z}} + \tag{6.3}$$

$$+ \frac{1}{\phi(w)|_{x=\text{Re } z, y=\text{Im } z}} - \phi(w)^{-1}|_{x=\text{Re } z, y=\text{Im } z}. \tag{6.4}$$

The expression in (6.3) can be written as

$$\frac{1}{\phi(z) \cdot \phi(w)|_{x=\text{Re } z, y=\text{Im } z}} \cdot (\phi(z) - \phi(w)|_{x=\text{Re } z, y=\text{Im } z}).$$

Here $\frac{1}{\phi(z)}$ is bounded by assumption. The assumed invertibility of $\phi(w)$ implies the boundedness $\phi(w)|_{x=\text{Re } z, y=\text{Im } z}$ on a certain neighbourhood of w . From $\phi \in \mathcal{F}_N$ we then conclude that (6.3) is a $O(\chi_w(z))$ for $z \rightarrow w$.

(6.4) can be rewritten as

$$- \frac{1}{\phi(w)|_{x=\text{Re } z, y=\text{Im } z}} \cdot \left(\phi(w)|_{x=\text{Re } z, y=\text{Im } z} \cdot \phi(w)^{-1}|_{x=\text{Re } z, y=\text{Im } z} - 1 \right).$$

The product in the brackets is a representative of $\phi(w) \cdot \phi(w)^{-1} = 1 + (P(w) \cdot Q(w)) \in \mathcal{A}(w)$. Hence, (6.4) equals to $\frac{1}{\phi(w)|_{x=\text{Re } z, y=\text{Im } z}} q(\text{Re } z, \text{Im } z)$ for a $q \in (P(w) \cdot Q(w))$, and is therefore a $O(\chi_w(z))$ for $z \rightarrow w$. Altogether (6.2) is a $O(\chi_w(z))$ for $z \rightarrow w$. Thus, $\phi^{-1} \in \mathcal{F}_N$. \square

7 Functional Calculus for zero-dimensional I

7.1 Lemma. *For each $\phi \in \mathcal{F}_N$ there exists $p \in \mathbb{C}[x, y]$ and complex valued $f_1, \dots, f_m \in \mathfrak{B}(\sigma(\Theta(N)) \cup V_{\mathbb{R}}(I))$ with $f_j(z) = 0$ for $z \in V_{\mathbb{R}}(I)$ such that*

$$\phi(z) = p_N(z) + \sum_j f_j(z) (p_j)_N(z)$$

for all $z \in \sigma(\Theta(N)) \cup V_{\mathbb{R}}(I)$, and that $\phi((\xi, \eta)) = p_N((\xi, \eta))$ for all $(\xi, \eta) \in V(I) \setminus \mathbb{R}^2$.

Proof. We apply Proposition 5.4 to $W = V_{\mathbb{R}}(I)$. The fact, that ψ is an isomorphism, then yields the existence of a polynomial $p \in \mathbb{C}[x, y]$ such that $p + (P(w) \cdot Q(w)) = \phi(w)$ for all $w \in V_{\mathbb{R}}(I)$ and such that $p + Q((\xi, \eta)) = \phi((\xi, \eta))$ for all $(\xi, \eta) \in V(I) \setminus \mathbb{R}^2$.

By Remark 6.4 we have $\phi(w) = p + (P(w) \cdot Q(w)) = p_N(w) \in \mathcal{A}(w)$ for $w \in V_{\mathbb{R}}(I)$. For $(\xi, \eta) \in V(I) \setminus \mathbb{R}^2$ we have $\phi((\xi, \eta)) = p + Q((\xi, \eta)) = p_N((\xi, \eta)) \in \mathcal{B}((\xi, \eta))$.

For $j = 1, \dots, m$ we set $f_j(z) := \frac{\phi(z) - p(z)}{\sum_k p_k(z)}$ if $z \in \sigma(\Theta(N)) \setminus V_{\mathbb{R}}(I)$ (see Lemma 3.5), and $f_j(z) = 0$ if $z \in V_{\mathbb{R}}(I)$. On $\sigma(\Theta(N)) \cup V_{\mathbb{R}}(I)$ we then have

$$\phi(z) = p_N(z) + \sum_j f_j(z) (p_j)_N(z).$$

It remains to verify that the functions f_j are measurable and bounded on $\sigma(\Theta(N)) \setminus V_{\mathbb{R}}(I)$. The measurability easily follows from the definition of f_j and the measurability of ϕ on this set. Since there are only finitely many points in $V_{\mathbb{R}}(I)$, the measurability of f_j on $\sigma(\Theta(N)) \cup V_{\mathbb{R}}(I)$ follows.

Concerning boundedness, note that by Lemma 6.10 $\phi - p_N$ belongs to \mathcal{F}_N . Since any representative $(\phi - p_N)(w)|_{x=\operatorname{Re} z, y=\operatorname{Im} z}$ of $(\phi - p_N)(w) \in \mathcal{A}(w)$ belongs to $P(w) \cdot Q(w) \subseteq Q(w)$, we have $(\phi - p_N)(z) = O(\chi_w(z))$ as $z \rightarrow w$ for any $w \in \sigma(\Theta(N)) \cap V_{\mathbb{R}}(I)$ which is not isolated on $\sigma(\Theta(N))$. By Lemma 3.5 we have $\chi_w(z) = O(\sum_k p_k(z))$ as $z \rightarrow w$ for $z \in \sigma(\Theta(N)) \setminus V_{\mathbb{R}}(I)$. Therefore,

$$f_j(z) = \frac{\phi(z) - p(z)}{\sum_k p_k(z)} = O(1) \quad \text{as } z \rightarrow w$$

for $z \in \sigma(\Theta(N)) \setminus V_{\mathbb{R}}(I)$. □

7.2 Definition. Let Δ be the set of all pairs $(\phi; (p, f_1|_{\sigma(\Theta(N))}, \dots, f_m|_{\sigma(\Theta(N))}))$ such that all assertions from Lemma 7.1 hold true for ϕ and (p, f_1, \dots, f_m) . ◇

7.3 Remark. It is straight forward to check that Δ is a linear subspace of $\mathcal{F}_N \times (\mathbb{C}[x, y] \times \mathfrak{B}(\sigma(\Theta(N))) \times \dots \times \mathfrak{B}(\sigma(\Theta(N))))$, i.e. a linear relations. Moreover, it is easy to check that with $(\phi; (p, f_1|_{\sigma(\Theta(N))}, \dots, f_m|_{\sigma(\Theta(N))}))$ also $(\phi^\#; (p^\#, \overline{f_1|_{\sigma(\Theta(N))}}, \dots, \overline{f_m|_{\sigma(\Theta(N))}}))$ belongs to Δ ; see Remark 4.5. ◇

Δ is also compatible with multiplication as will be shown next.

7.4 Lemma. *If both, $(\phi; (p, f_1|_{\sigma(\Theta(N))}, \dots, f_m|_{\sigma(\Theta(N))}))$ and $(\psi; (q, g_1|_{\sigma(\Theta(N))}, \dots, g_m|_{\sigma(\Theta(N))}))$, belong to Δ , then also the pair $(\phi \cdot \psi; (r, h_1|_{\sigma(\Theta(N))}, \dots, h_m|_{\sigma(\Theta(N))}))$ belongs to Δ , where (see Definition 4.8)*

$$(r, h_1|_{\sigma(\Theta(N))}, \dots, h_m|_{\sigma(\Theta(N))}) = (p, f_1|_{\sigma(\Theta(N))}, \dots, f_m|_{\sigma(\Theta(N))}) \cdot (q, g_1|_{\sigma(\Theta(N))}, \dots, g_m|_{\sigma(\Theta(N))}).$$

Proof. On $\sigma(\Theta(N)) \cup V_{\mathbb{R}}(I)$ we have

$$\phi(z) = p_N(z) + \sum_j f_j(z)(p_j)_N(z) \quad \text{and} \quad \psi(z) = q_N(z) + \sum_j g_j(z)(p_j)_N(z).$$

Moreover, $f_j(z) = 0 = g_j$ for $z \in V_{\mathbb{R}}(I)$, and $\phi((\xi, \eta)) = p_N((\xi, \eta))$, $\psi((\xi, \eta)) = q_N((\xi, \eta))$ for all $(\xi, \eta) \in V(I) \setminus \mathbb{R}^2$.

Since $p \mapsto p_N$ is compatible with multiplication, for $r = p \cdot q$ we have $(\phi \cdot \psi)((\xi, \eta)) = r_N((\xi, \eta))$ for all $(\xi, \eta) \in V(I) \setminus \mathbb{R}^2$. Clearly, $h_j = pg_j + qf_j + f_j \sum_{k=1}^m g_k p_k$ vanishes on $V_{\mathbb{R}}(I)$. For $z \in \sigma(\Theta(N)) \cup V_{\mathbb{R}}(I)$ we have

$$\begin{aligned} \phi(z) \psi(z) &= p_N(z) q_N(z) + \\ &\quad \sum_j \left(p_N(z) g_j(z) + q_N(z) f_j(z) + f_j(z) \sum_k g_k(z) (p_k)_N(z) \right) (p_j)_N(z), \end{aligned}$$

which, for $z \in V_{\mathbb{R}}(I)$, coincides with $r_N(z) = r_N(z) + \sum_j h_j(z)(p_j)_N(z)$. For $z \in \sigma(\Theta(N)) \setminus V_{\mathbb{R}}(I)$ the above equation can be written as

$$\begin{aligned} \phi(z) \psi(z) &= r(z) + \sum_j \left(p(z) g_j(z) + q(z) f_j(z) + f_j(z) \sum_k g_k(z) p_k(z) \right) p_j(z) \\ &= r_N(z) + \sum_j h_j(z) (p_j)_N(z). \end{aligned}$$

□

We are going to determine the multivalued part $\text{mul } \Delta$ of Δ .

7.5 Lemma. *Assume that $p(x, y) \in \mathbb{C}[x, y]$ and $f_1, \dots, f_m \in \mathfrak{B}(\sigma(\Theta(N)) \cup V_{\mathbb{R}}(I))$ with $f_j(z) = 0$ for $z \in V_{\mathbb{R}}(I)$ such that*

$$0 = p_N(z) + \sum_j f_j(z)(p_j)_N(z)$$

on $\sigma(\Theta(N)) \cup V_{\mathbb{R}}(I)$ and that $\phi((\xi, \eta)) = 0$ for all $(\xi, \eta) \in V(I) \setminus \mathbb{R}^2$. Then $(p, f_1|_{\sigma(\Theta(N))}, \dots, f_m|_{\sigma(\Theta(N))})$ belongs to the ideal \mathcal{N} in \mathcal{R} as defined in Definition 4.4.

Proof. Clearly, $p + \sum_{j=1}^m f_j p_j = 0$ on $\sigma(\Theta(N)) \setminus V_{\mathbb{R}}(I)$.

According to Remark 6.4 $p + (P(w) \cdot Q(w)) = 0 \in \mathcal{A}(w)$ for all $w \in V_{\mathbb{R}}(I)$ and $p + Q((\xi, \eta)) = 0 \in \mathcal{B}((\xi, \eta))$ for all $(\xi, \eta) \in V(I) \setminus \mathbb{R}^2$. Hence, $p \in \bigcap_{(\xi, \eta) \in V(I) \setminus \mathbb{R}^2} Q((\xi, \eta)) \cap \bigcap_{w \in V_{\mathbb{R}}(I)} (P(w) \cdot Q(w))$. By Proposition 5.4 we therefore have $p = \sum_j u_j p_j$ with $u_j(w) = 0$ for all $w \in V_{\mathbb{R}}(I)$. We see that $(f_j + u_j)(z) = 0$ for all $z \in V_{\mathbb{R}}(I) \cap \sigma(\Theta(N))$. Thus, $(p, f_1|_{\sigma(\Theta(N))}, \dots, f_m|_{\sigma(\Theta(N))}) \in \mathcal{N}$. □

Since by Lemma 4.6 $\text{mul } \Delta \subseteq \mathcal{N} \subseteq \ker \Psi$ the composition $\Psi \Delta$ is a well-defined linear mapping from \mathcal{F}_N into $B(\mathcal{K})$.

7.6 Definition. For $\phi \in \mathcal{F}_N$ we set $\phi(N) := (\Psi \Delta)(\phi)$. \diamond

By Theorem 4.10, Lemma 7.4 and Remark 7.3 the following result can be formulated.

7.7 Theorem. $\phi \mapsto \phi(N)$ constitutes a $*$ -homomorphism from \mathcal{F}_N into $\{N, N^*\}'' \subseteq B(\mathcal{K})$. It satisfies $p_N(N) = p(A, B)$ for all $p \in \mathbb{C}[x, y]$.

Proof. The final assertion is clear because of $(p_N; (p, 0, \dots, 0)) \in \Delta$. \square

8 Spectral properties of the functional calculus

For $w \in V_{\mathbb{R}}(I)$ we will need the following notation. By $\pi_w : \mathcal{A}(w) \rightarrow \mathcal{B}(w)$ we denote the mapping

$$\pi_w(f + (P(w) \cdot Q(w))) = f + Q(w).$$

8.1 Lemma. If $\phi \in \mathcal{F}_N$ vanishes everywhere except at a fixed $w \in V_{\mathbb{R}}(I)$ and if $\pi_w \phi(w) = 0$, then

$$\phi(N) = \Psi(0; g_1, \dots, g_m)$$

for $g_1, \dots, g_m \in \mathfrak{B}(\sigma(\Theta(N)))$ which vanish on $(\sigma(\Theta(N)) \cup V_{\mathbb{R}}(I)) \setminus \{w\}$.

Proof. Let $p(x, y) \in \mathbb{C}[x, y]$ and $f_1, \dots, f_m \in \mathfrak{B}(\sigma(\Theta(N)) \cup V_{\mathbb{R}}(I))$ with $f_j(z) = 0$ for $z \in V_{\mathbb{R}}(I)$ such that

$$\phi(z) = p_N(z) + \sum_j f_j(z) (p_j)_N(z)$$

for all $z \in \sigma(\Theta(N)) \cup V_{\mathbb{R}}(I)$, and that $p_N((\xi, \eta)) = \phi((\xi, \eta)) = 0$ for all $(\xi, \eta) \in V(I) \setminus \mathbb{R}^2$. The latter fact just means $p \in p((\xi, \eta)) \in Q((\xi, \eta))$. From $0 = \phi(z) = p_N(z) + \sum_j f_j(z) (p_j)_N(z)$ for $z \in V_{\mathbb{R}}(I) \setminus \{w\}$ we infer $p \in (P(z) \cdot Q(z))$. For $z = w$ this equation together with $\pi_w \phi(w) = 0$ yields $p \in Q(w)$.

By Proposition 5.4 $p = \sum_j u_j p_j$, where $u_j(z) = 0$ for all $z \in V_{\mathbb{R}}(I) \setminus \{w\}$. We define g_j to be zero on $(\sigma(\Theta(N)) \cup V_{\mathbb{R}}(I)) \setminus \{w\}$ and set $g_j(w) = u_j(w)$. The difference

$$(p; f_1|_{\sigma(\Theta(N))}, \dots, f_m|_{\sigma(\Theta(N))}) - (0; g_1, \dots, g_m) =$$

$$(p; f_1|_{\sigma(\Theta(N))} - \delta_w(\cdot)u_1(w), \dots, f_m|_{\sigma(\Theta(N))} - \delta_w(\cdot)u_m(w))$$

satisfies $p + \sum_j (f_j(z) - \delta_w(z)u_j(w))p_j(z) = \phi(z) = 0$ for $z \in \sigma(\Theta(N)) \setminus V_{\mathbb{R}}(I)$ and $f_j(z) - \delta_w(z)u_j(w) + u_j(z) = 0$ for all $z \in V_{\mathbb{R}}(I) \cap \sigma(\Theta(N))$. It therefore belongs to the ideal \mathcal{N} of \mathcal{R} . Consequently,

$$\phi(N) = \Psi(p; f_1|_{\sigma(\Theta(N))}, \dots, f_m|_{\sigma(\Theta(N))}) = \Psi(0; g_1, \dots, g_m).$$

\square

8.2 Corollary. Assume that $E\{w\} = 0$ for a fixed $w \in V_{\mathbb{R}}(I)$, which surely happens if $w \notin \sigma(\Theta(N))$. Then $\phi(N) = \psi(N)$ for all ϕ, ψ that coincide on $((\sigma(\Theta(N)) \cup V_{\mathbb{R}}(I)) \setminus \{w\}) \dot{\cup} (V(I) \setminus \mathbb{R}^2)$ and that satisfy $\pi_w \phi(w) = \pi_w \psi(w)$. Here $\pi_w : \mathcal{A}(w) \rightarrow \mathcal{B}(w)$ is defined by $\pi_w(f + (P(w) \cdot Q(w))) = f + Q(w)$.

Proof. By Lemma 8.1 there exist $g_1, \dots, g_m \in \mathfrak{B}(\sigma(\Theta(N)))$, which vanish on $(\sigma(\Theta(N)) \cup V_{\mathbb{R}}(I)) \setminus \{w\}$, such that

$$\phi(N) - \psi(N) = \Psi(0; g_1, \dots, g_m) = \sum_{k=1}^m \Xi_k \left(\int_{\sigma(\Theta_k(N))} g_k dE_k \right)$$

According to Lemma 4.1 together with our assumption $E\{w\} = 0$, this operator vanishes. \square

8.3 Remark. For $\zeta \in V(I) \setminus \mathbb{R}^2$ or a $\zeta \in V_{\mathbb{R}}(I)$, which is isolated in $\sigma(\Theta(N)) \cup V_{\mathbb{R}}(I)$, we saw in Example 6.8 that $a\delta_{\zeta} \in \mathcal{F}_N$. If a is the unite e in $\mathcal{B}(\zeta)$ or in $\mathcal{A}(\zeta)$, i.e. the coset $1 + Q(\zeta)$ for $\zeta \in V(I) \setminus \mathbb{R}^2$ or the coset $1 + (P(\zeta) \cdot Q(\zeta))$ for $\zeta \in V_{\mathbb{R}}(I)$, then $(e\delta_{\zeta}) \cdot (e\delta_{\zeta}) = (e\delta_{\zeta})$ together with the multiplicativity of $\phi \mapsto \phi(N)$ shows that $(e\delta_{\zeta})(N)$ is a projection. It is a kind of Riesz projection corresponding to ζ .

We set $\xi := \operatorname{Re} \zeta$, $\eta := \operatorname{Im} \zeta$ if $\zeta \in V_{\mathbb{R}}(I)$ and $(\xi, \eta) := \zeta$ if $\zeta \in V(I) \setminus \mathbb{R}^2$. For $\lambda \in \mathbb{C} \setminus \{\xi + i\eta\}$ and for $s(z, w) := z + iw - \lambda$ we then have $s_N \cdot (e\delta_{\zeta}) = (s_N(\zeta))\delta_{\zeta}$. As $s(\xi, \eta) \neq 0$, $s_N(\zeta)$ does not belong to $P(\zeta) \supseteq Q(\zeta)$. Therefore, it is invertible in $\mathcal{B}(\zeta)$ or in $\mathcal{A}(\zeta)$. For its inverse b we obtain

$$s_N \cdot (e\delta_{\zeta}) \cdot (b\delta_{\zeta}) = e\delta_{\zeta}.$$

From $s_N(N) = N - \lambda$ we derive that $(N|_{\operatorname{ran}(e\delta_{\zeta})(N)} - \lambda)^{-1} = (b\delta_{\zeta})(N)|_{\operatorname{ran}(e\delta_{\zeta})(N)}$ on $\operatorname{ran}(e\delta_{\zeta})(N)$. In particular, $\sigma(N|_{\operatorname{ran}(e\delta_{\zeta})(N)}) \subseteq \{\xi + i\eta\}$. \diamond

8.4 Lemma. *If $\phi \in \mathcal{F}_N$ vanishes on*

$$(\sigma(\Theta(N)) \cup (V_{\mathbb{R}}(I) \cap \sigma(N))) \dot{\cup} \{(\alpha, \beta) \in V(I) \setminus \mathbb{R}^2 : \alpha + i\beta, \bar{\alpha} + i\bar{\beta} \in \sigma(N)\},$$

then $\phi(N) = 0$.

Proof. Since any $w \in V_{\mathbb{R}}(I) \setminus \sigma(N)$ is isolated in $\sigma(\Theta(N)) \cup V_{\mathbb{R}}(I)$, we saw in Remark 8.3 that for

$$\zeta \in \underbrace{(V_{\mathbb{R}}(I) \setminus \sigma(N))}_{=: Z_1} \dot{\cup} \underbrace{\{(\alpha, \beta) \in V(I) \setminus \mathbb{R}^2 : \alpha + i\beta \in \rho(N)\}}_{=: Z_2}$$

the expression $(e\delta_{\zeta})(N)$ is a bounded projection commuting with N . Hence, $(e\delta_{\zeta})(N)$ also commutes with $(N - (\xi + i\eta))^{-1}$, where $\xi := \operatorname{Re} \zeta$, $\eta := \operatorname{Im} \zeta$ if $\zeta \in Z_1$ and $(\xi, \eta) := \zeta$ if $\zeta \in Z_2$.

Consequently, $N|_{\operatorname{ran}(e\delta_{\zeta})(N)} - (\xi + i\eta)$ is invertible on $\operatorname{ran}(e\delta_{\zeta})(N)$, i.e. $\xi + i\eta \notin \sigma(N|_{\operatorname{ran}(e\delta_{\zeta})(N)})$. In Remark 8.3 we saw $\sigma(N|_{\operatorname{ran}(e\delta_{\zeta})(N)}) \subseteq \{\xi + i\eta\}$. Hence, $\sigma(N|_{\operatorname{ran}(e\delta_{\zeta})(N)}) = \emptyset$, which is impossible for $\operatorname{ran}(e\delta_{\zeta})(N) \neq \{0\}$. Thus, $(e\delta_{\zeta})(N) = 0$.

For $(\xi, \eta) \in Z_3 := \{(\alpha, \beta) \in V(I) \setminus \mathbb{R}^2 : \bar{\alpha} + i\bar{\beta} \in \rho(N)\}$ one has $(\bar{\xi}, \bar{\eta}) \in Z_2$. Hence,

$$0 = (e\delta_{(\bar{\xi}, \bar{\eta})})(N)^* = (e^{\#}\delta_{(\xi, \eta)})(N) = (e\delta_{(\xi, \eta)})(N).$$

Since, by our assumption, ϕ is supported on $Z_1 \cup Z_2 \cup Z_3$, we obtain

$$\phi(N) = \left(\sum_{\zeta \in Z_1 \cup Z_2 \cup Z_3} \phi(\zeta)\delta_{\zeta} \right)(N) = \sum_{\zeta \in Z_1 \cup Z_2 \cup Z_3} \phi(\zeta)(e\delta_{\zeta})(N) = 0.$$

\square

As a consequence of Lemma 8.4 for $\phi \in \mathcal{F}_N$ the operator $\phi(N)$ only depends on ϕ 's values on

$$\begin{aligned} & (\sigma(\Theta(N)) \cup (V_{\mathbb{R}}(I) \cap \sigma(N))) \dot{\cup} \\ & \{(\alpha, \beta) \in V(I) \setminus \mathbb{R}^2 : \alpha + i\beta, \bar{\alpha} + i\bar{\beta} \in \sigma(N)\}. \end{aligned} \quad (8.1)$$

Thus, we can, and will from now on, re-define the function class \mathcal{F}_N for our functional calculus so that the elements ϕ of \mathcal{F}_N are functions on this set with values in $\mathbb{C}, \mathcal{A}(z)$ or $\mathcal{B}(z)$, such that $z \mapsto \phi(z)$ is measurable and bounded on $\sigma(\Theta(N)) \setminus V_{\mathbb{R}}(I)$ and such that (6.1) holds true for every $w \in \sigma(\Theta(N)) \cap V_{\mathbb{R}}(I)$ which is not isolated in $\sigma(\Theta(N))$.

8.5 Lemma. *If $\phi \in \mathcal{F}_N$ is such that $\phi(z)$ is invertible in $\mathbb{C}, \mathcal{A}(z)$ or $\mathcal{B}(z)$, respectively, for all z in (8.1), and such that 0 does not belong to the closure of $\phi(\sigma(\Theta(N)) \setminus V_{\mathbb{R}}(I))$, then $\phi(N)$ is a boundedly invertible operator on \mathcal{K} with $\phi^{-1}(N)$ as its inverse.*

Proof. We think of ϕ as a function on $(\sigma(\Theta(N)) \cup V_{\mathbb{R}}(I)) \dot{\cup} (V(I) \setminus \mathbb{R}^2)$ by setting $\phi(z) = e$ for all z not belonging to (8.1). Then all assumptions of Lemma 6.11 are satisfied. Hence $\phi^{-1} \in \mathcal{F}_N$, and we conclude from Theorem 7.7 and Remark 6.4 that

$$\phi^{-1}(N)\phi(N) = \phi(N)\phi^{-1}(N) = (\phi \cdot \phi^{-1})(N) = \mathbb{1}_N(N) = I_{\mathcal{K}}.$$

□

8.6 Corollary. $\sigma(N)$ equals to

$$\begin{aligned} & \sigma(\Theta(N)) \cup (V_{\mathbb{R}}(I) \cap \sigma(N)) \cup \\ & \{\alpha + i\beta : (\alpha, \beta) \in V(I) \setminus \mathbb{R}^2, \alpha + i\beta, \bar{\alpha} + i\bar{\beta} \in \sigma(N)\}. \end{aligned} \quad (8.2)$$

In particular, $\sigma(N) \setminus \sigma(\Theta(N))$ is finite.

Proof. Since Θ is a homomorphism, we have $\sigma(\Theta(N)) \subseteq \sigma(N)$. Hence, (8.2) is contained in $\sigma(N)$. For the converse, consider the polynomial $s(z, w) = z + iw - \lambda$ for a λ not belonging to (8.2). We conclude that for any

$$\zeta \in (V_{\mathbb{R}}(I) \cap \sigma(N)) \cup \{(\alpha, \beta) \in V(I) \setminus \mathbb{R}^2 : \alpha + i\beta, \bar{\alpha} + i\bar{\beta} \in \sigma(N)\}$$

the polynomial s does not belong to $P(\zeta) \supseteq Q(\zeta)$. Hence, $s_N(\zeta)$ is invertible $\mathcal{A}(\zeta)$ or $\mathcal{B}(\zeta)$. Clearly, $s_N(\zeta) \neq 0$ for $\zeta \in \sigma(\Theta(N)) \setminus V_{\mathbb{R}}(I)$. Finally, 0 does not belong to the closure of

$$s_N(\sigma(\Theta(N)) \setminus V_{\mathbb{R}}(I)) = s(\sigma(\Theta(N)) \setminus V_{\mathbb{R}}(I)) \subseteq \sigma(\Theta(N)) - \lambda.$$

Applying Lemma 8.5, we see that $s_N(N) = (N - \lambda)$ is invertible. □

8.7 Remark. We set $K_r := V_{\mathbb{R}}(I) \cap \sigma(N)$,

$$Z := \{(\alpha, \beta) \in V(I) \setminus \mathbb{R}^2 : \alpha + i\beta, \bar{\alpha} + i\bar{\beta} \in \sigma(N)\},$$

and $K_i := \{\alpha + i\beta : (\alpha, \beta) \in Z\}$. Using Corollary 8.6 we could re-define once more the functions $\phi \in \mathcal{F}_N$ as functions ϕ on $\sigma(N)$ such that

1. ϕ is complex valued, bounded and measurable on $\sigma(N) \setminus (K_r \cup K_i)$,
2. $\phi(\zeta) \in \mathcal{A}(\zeta)$ for $\zeta \in K_r \setminus K_i$,
3. $\phi(\zeta) \in \times_{(\alpha, \beta) \in Z, \alpha + i\beta = \zeta} \mathcal{A}(\zeta)$ for $\zeta \in K_i \setminus K_r$,
4. $\phi(\zeta) \in \mathcal{A}(\zeta) \times \times_{(\alpha, \beta) \in Z, \alpha + i\beta = \zeta} \mathcal{A}(\zeta)$ for $\zeta \in K_r \cap K_i$;
5. for a $w \in K_r$, which is not isolated in $\sigma(N)$, we have

$$\phi(z) - p(\operatorname{Re} z, \operatorname{Im} z) = O(\chi_w(z)) \quad \text{as} \quad \sigma(N) \setminus (K_r \cup K_i) \ni z \rightarrow w,$$

where p is a representative of $\phi(w)$ for $w \in K_r \setminus K_i$ and p is a representative of the first entry of $\phi(w)$ for $w \in K_r \cap K_i$.

◇

9 Special cases of definitizable operators

Unitary and selfadjoint operators are special cases of normal operators on Hilbert spaces as well as on Krein spaces. We will show how some well-known facts on definitizable selfadjoint or unitary operators on a Krein space \mathcal{K} can easily be obtained from the previously obtained results.

9.1 Selfadjoint definitizable operators

An operator $N \in B(\mathcal{K})$ is by definition selfadjoint if $N = N^+$. Obviously, $N \in B(\mathcal{K})$ is selfadjoint if and only if N is normal and satisfies $p(A, B) = 0$, where $A = \frac{N+N^+}{2}$, $B = \frac{N-N^+}{2i}$ and

$$p(x, y) = y \in \mathbb{R}[x, y].$$

Therefore, according to Definition 3.1 any selfadjoint operator on a Krein space is definitizable normal, and the ideal I generated by all real definitizing polynomials contains $p(x, y) = y$. Since the ideal generated by $p(x, y) = y$ is not zero-dimensional, the zero-dimensionality of I implies the existence of at least one real definitizing polynomial of the form

$$y \cdot s(x, y) + t(x) \quad \text{with} \quad s(x, y) \in \mathbb{C}[z, w], \quad t(x) \in \mathbb{C}[x] \setminus \{0\}. \quad (9.1)$$

9.1 Proposition. *The ideal I is zero-dimensional if and only if there exists a $t \in \mathbb{R}[x] \setminus \{0\}$ such that $[t(A)u, u] \geq 0$, $u \in \mathcal{K}$, i.e. $N = A$ is definitizable in the classical sense; see [KP].*

Proof. Any $r(x, y) \in \mathbb{C}[z, w]$ can be written as $r(x, y) = y \cdot s_r(x, y) + t_r(x)$ with unique $s_r(x, y) \in \mathbb{C}[z, w]$, $t_r(x) \in \mathbb{C}[x]$. Hence, $r \in I$ if and only if $t_r(x) \in I$. The set of $I_x := \{t_r : r \in I\}$ forms an ideal in $\mathbb{C}[x]$. If I_x is the zero ideal, then $I = y \cdot \mathbb{C}[x, y]$ is not zero-dimensional.

If $I_x \neq \{0\}$, then, applying the polynomial division, we see that $\dim \mathbb{C}[x]/I_x < \infty$. This implies the zero-dimensionality of I . If $r(x, y)$ is a real definitizing polynomial as in (9.1), then

$$[t(A)u, u] = [r(A, B)u, u] \geq 0, \quad u \in \mathcal{K},$$

i.e. $t(x)$ is a definitizing polynomial. □

Assume that $N \in B(\mathcal{K})$ is selfadjoint and that the ideal I generated by all real definitizing polynomials is zero-dimensional. Consequently, we can apply the functional calculus developed in Section 7. From $p(x, y) = y \in I$ we conclude

$$a = (a_x, a_y) \in V(I) \Rightarrow a_y = p(a) = 0.$$

Hence, the elements of $V_{\mathbb{R}}(I)$ are contained in \mathbb{R} , and $(\xi, \eta) \in V(I) \setminus \mathbb{R}^2$ yields $\eta = 0$. Moreover, with N also $\Theta(N)$ is selfadjoint in the Hilbert space \mathcal{H} ; see Proposition 3.3 and (2.1). In particular, $\sigma(\Theta(N)) \subseteq \mathbb{R}$. From Corollary 8.6 we derive that $\sigma(N)$ is contained in \mathbb{R} up to finitely many points which are located in $\mathbb{C} \setminus \mathbb{R}$ symmetric with respect to \mathbb{R} .

9.2 Unitary definitizable operators

An operator $N \in B(\mathcal{K})$ is by definition unitary if $N^+N = NN^+ = I_{\mathcal{K}}$. Obviously, $N \in B(\mathcal{K})$ is unitary if and only if N is normal and satisfies $p(A, B) = 0$, where $A = \frac{N+N^+}{2}$, $B = \frac{N-N^+}{2i}$ and

$$p(x, y) = (x + iy)(x - iy) - 1 = x^2 + y^2 - 1 \in \mathbb{R}[x, y].$$

Therefore, according to Definition 3.1 any unitary operator on a Krein space is definitizable normal, and the ideal I generated by all real definitizing polynomials always contains $p(x, y)$. Since the ideal generated by p is not zero-dimensional, the zero-dimensionality of I implies the existence a definitizing polynomial different from p .

9.2 Remark. If, for example, there exists a polynomial $a \in \mathbb{C}[z] \setminus \{0\}$ such that $[a(N)u, u] \geq 0$, $u \in \mathcal{K}$, then the ideal J generated by a (considered as a polynomial in $\mathbb{C}[z, w]$) and $b(z, w) = zw - 1$ in $\mathbb{C}[z, w]$ is zero-dimensional. Indeed, it is easy to see that the set $V(J)$ of common zeros of a and b is finite, which by [CLO2], page 39, implies zero-dimensionality. Since $c(z, w) \mapsto c(x + iy, x - iy)$ constitutes an isomorphism from $\mathbb{C}[z, w]$ onto $\mathbb{C}[x, y]$, also the ideal generated by $a(x + iy)$ and $p(x, y)$ in $\mathbb{C}[x, y]$ is zero-dimensional. Hence, the same is true for I , and we can apply the functional calculus developed Section 7. \diamond

Assume that $N \in B(\mathcal{K})$ is unitary and that the ideal I generated by all real definitizing polynomials is zero-dimensional. Consequently, we can apply the functional calculus developed in Section 7. From $p \in I$ we conclude

$$a \in V(I) \Rightarrow p(a) = 0.$$

Hence, the elements of $V_{\mathbb{R}}(I)$ are contained in \mathbb{T} , and $(\xi, \eta) \in V(I) \setminus \mathbb{R}^2$ yields

$$(\xi + i\eta)(\overline{\xi + i\eta}) = \xi^2 + \eta^2 = 1.$$

Moreover, with N also $\Theta(N)$ is unitary in the Hilbert space \mathcal{H} ; see Proposition 3.3 and (2.1). In particular, $\sigma(\Theta(N)) \subseteq \mathbb{T}$. From Corollary 8.6 we derive that $\sigma(N)$ is contained in \mathbb{T} up to finitely many points which are located in $\mathbb{C} \setminus \mathbb{T}$ symmetric with respect to \mathbb{T} .

10 Transformations of definitizable normal operators

In this final section we examine, whether basic transformations, such as $\alpha N, N + \beta I_K, N^{-1}$ with $\alpha, \beta \in \mathbb{C}$, $\alpha \neq 0$, of definitizable normal operators N are again definitizable, and how the corresponding ideals I behave.

For $\beta \in \mathbb{C}$ it is easy to see that $p(x, y)$ is a real definitizing polynomial for N if and only if $p(x - \operatorname{Re} \beta, y - \operatorname{Im} \beta)$ is real definitizing for $N + \beta I_K$. Since $r(x, y) \mapsto r(x - \operatorname{Re} \beta, y - \operatorname{Im} \beta)$ is a ring automorphism on $\mathbb{C}[x, y]$, the respective ideals I , corresponding to N and $N + \beta I_K$, are zero-dimensional, or not, at the same time.

Similarly, $p(x, y)$ is a real definitizing polynomial for N if and only if $p(x \operatorname{Re} \frac{1}{\alpha} - y \operatorname{Im} \frac{1}{\alpha}, x \operatorname{Im} \frac{1}{\alpha} + y \operatorname{Re} \frac{1}{\alpha})$ is real definitizing for αN . Also $r(x, y) \mapsto r(x \operatorname{Re} \frac{1}{\alpha} - y \operatorname{Im} \frac{1}{\alpha}, x \operatorname{Im} \frac{1}{\alpha} + y \operatorname{Re} \frac{1}{\alpha})$ is a ring automorphism on $\mathbb{C}[x, y]$. Hence, the ideal I corresponding to N is zero-dimensional if and only if the ideal I corresponding to αN is zero-dimensional.

For the inverse N^{-1} the situation is more complicated. We formulate two results that we will need. The first assertion is straight forward to verify. We omit its proof.

10.1 Lemma. *The mapping $\Phi : p(x, y) \mapsto p(\frac{z+w}{2}, \frac{z-w}{2i})$ from $\mathbb{C}[x, y]$ to $\mathbb{C}[z, w]$ is an isomorphism, where p is real, i.e. $p(\bar{x}, \bar{y}) = \overline{p(x, y)}$, if and only if $\Phi(p)(z, w) = \Phi(p)(\bar{w}, \bar{z})$.*

Obviously, for a normal $N = A + iB$ and $p(x, y) \in \mathbb{C}[x, y]$ we have

$$p(A, B) = \Phi(p)(N, N^+). \quad (10.1)$$

For a polynomial $q \in \mathbb{C}[z, w] \setminus \{0\}$ let $d(q)$ be the maximum of the z -degree of q and the w -degree of q . Moreover, we set

$$\varpi(q)(z, w) := (zw)^{d(q)} q\left(\frac{1}{z}, \frac{1}{w}\right) \in \mathbb{C}[z, w].$$

10.2 Lemma. *If $I = \langle q_1, \dots, q_m \rangle$ is zero-dimensional with polynomials q_1, \dots, q_m such that $\overline{q_j(z, w)} = q_j(\bar{w}, \bar{z})$, then the ideal $\langle \varpi(q_1), \dots, \varpi(q_m) \rangle$ is also zero-dimensional.*

Proof. Let $(\zeta, \eta) \in V(\varpi(q_1), \dots, \varpi(q_m))$. For $\zeta \neq 0 \neq \eta$ we conclude $q_j(\frac{1}{\zeta}, \frac{1}{\eta}) = 0$, $j = 1, \dots, m$, and in turn $(\zeta, \eta) \in \{(z, w) \in (\mathbb{C} \setminus \{0\})^2 : (\frac{1}{z}, \frac{1}{w}) \in V(I)\}$.

Assume that $\eta = 0$ and $\zeta \neq 0$. If $q_j(z, w) = \sum_{k,l=0}^{d(q_j)} b_{k,l} z^k w^l$, then $\overline{q_j(z, w)} = q_j(\bar{w}, \bar{z})$ yields $b_{k,l} = \bar{b}_{l,k}$, and we have $\varpi(q_j)(z, w) = \sum_{k,l=0}^{d(q_j)} b_{d(q_j)-k, d(q_j)-l} z^k w^l$. According to the choice of $d(q_j)$ and by $b_{k,l} = \bar{b}_{l,k}$ the polynomial

$$\rho_j(z) := \varpi(q_j)(z, 0) = \sum_{k=0}^{d(q_j)} b_{d(q_j)-k, d(q_j)} z^k$$

is non-zero and satisfies $\rho_j(\zeta) = 0$, i.e. $(\zeta, \eta) \in \rho_j^{-1}(\{0\}) \times \{0\}$.

From $\overline{q_j(z, w)} = q_j(\bar{w}, \bar{z})$ we conclude $\rho_j(\bar{w}) = \overline{\varpi(q_j)(0, w)}$. Hence, $\zeta = 0$ and $\eta \neq 0$ yields $(\zeta, \eta) \in \{0\} \times \rho_j^{-1}(\{0\})$.

In any case (ζ, η) is contained in

$$\begin{aligned} & \{(0, 0)\} \cup \{(z, w) \in (\mathbb{C} \setminus \{0\})^2 : (\frac{1}{z}, \frac{1}{w}) \in V(I)\} \cup \\ & \cup \bigcap_{j=1, \dots, m} \rho_j^{-1}(\{0\}) \times \{0\} \cup \bigcap_{j=1, \dots, m} \{0\} \times \overline{\rho_j^{-1}(\{0\})}. \end{aligned}$$

Consequently, $V(\varpi(q_1), \dots, \varpi(r_m))$ is finite, and in turn $\langle \varpi(q_1), \dots, \varpi(r_m) \rangle$ is zero-dimensional; see [CLO2], page 39. \square

10.3 Proposition. *Let N be normal and bijective on the Krein space \mathcal{K} . If $p(x, y)$ is real definitizing for N , then $\Phi^{-1}(\varpi(\Phi(p)))$ is definitizing for N^{-1} . Moreover, if the ideal I generated by all real definitizing $p(x, y)$ for N is zero-dimensional, then also the ideal generated by all real definitizing polynomials for N^{-1} is zero-dimensional.*

Proof. Let $p(x, y)$ be real definitizing for N . By Lemma 10.1 we have $\overline{\Phi(p)(z, w)} = \Phi(p)(\bar{w}, \bar{z})$, and in turn $\overline{\varpi(\Phi(p))(z, w)} = \varpi(\Phi(p))(\bar{w}, \bar{z})$. We write $\Phi(p)(z, w) = \sum_{k, l=0}^{d(\Phi(p))} b_{k, l} z^k w^l$, and consequently $\varpi(\Phi(p))(z, w) = \sum_{k, l=0}^{d(\Phi(p))} b_{d(\Phi(p))-k, d(\Phi(p))-l} z^k w^l$.

By (10.1) for $u \in \mathcal{K}$ we have

$$\begin{aligned} & [\Phi^{-1}(\varpi(\Phi(p)))(\operatorname{Re} N^{-1}, \operatorname{Im} N^{-1})u, u] = [\varpi(\Phi(p))(N^{-1}, N^{-+})u, u] \\ & = [\sum_{k, l=0}^{d(\Phi(p))} b_{d(\Phi(p))-k, d(\Phi(p))-l} (N^{-1})^k (N^{-+})^l u, u] \\ & = [\Phi(p)(N, N^+) (N^{-1})^{d(\Phi(p))} u, (N^{-1})^{d(\Phi(p))} u] \\ & = [p(A, B) (N^{-1})^{d(\Phi(p))} u, (N^{-1})^{d(\Phi(p))} u] \geq 0. \end{aligned}$$

Hence, $\Phi^{-1}(\varpi(\Phi(p)))$ is real definitizing for N^{-1} . Finally, if I is zero-dimensional and generated by real definitizing p_1, \dots, p_m , then $\Phi(I) = \langle \Phi(p_1), \dots, \Phi(p_m) \rangle$ is zero-dimensional in $\mathbb{C}[z, w]$. According to Lemma 10.2 $\langle \varpi(\Phi(p_1)), \dots, \varpi(\Phi(p_m)) \rangle$, and hence also $\langle \Phi^{-1}(\varpi(\Phi(p_1))), \dots, \Phi^{-1}(\varpi(\Phi(p_m))) \rangle$ is zero-dimensional. Since its generators are real definitizing for N^{-1} also the ideal generated by all real definitizing polynomials for N^{-1} is zero-dimensional. \square

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